Quantum Noise Minimization in Transistor Amplifiers

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General quantum restrictions on the noise performance of linear transistor amplifiers are used to identify the region in parameter space where the quantum-limited performance is achievable and to construct a practical procedure for approaching it experimentally using only the knowledge of directly measurable quantities: the gain, (differential) conductance, and the output noise. A specific example of resonant barrier transistors is discussed.

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Heisenberg uncertainty relations restrict the performance of amplifiers and detectors [1–10]. Derived from rather general properties (canonical commutation relations for signals carried by nonconserved bosons [1] or the nonequilibrium Kubo formula for other signals [6–10]), such restrictions specify the best-possible noise performance but do not provide a procedure for obtaining it. For example, a (phase insensitive) linear amplifier must add to the amplified signal a noise power of at least \((G^2 - 1)\hbar \omega / 2\) per unit bandwidth [11], where \(G^2\) is the power gain [1,2,6,10]. This restriction, referred to below as the Heisenberg limit, is very general and applies, e.g., to laser amplifiers, parametric rf amplifiers, field effect transistors, single-electron transistors, and molecular transistors. However, the particular source of the noise varies and, therefore, also the procedures one needs to follow in order to minimize it. In parametric amplifiers, this noise is the equilibrium current noise in the idler resistor [2], and, therefore, this resistor should be cold enough to produce only the zero point fluctuations.

In transistor devices, in which the amplification is performed by a signal on a gate strongly modulating the output current, cooling the device is not sufficient to obtain the ideal noise performance. Such devices manifest nonequilibrium noise (called idling noise below) in the source-drain current even when the gate voltage is held fixed. When the gate is connected to a signal source having nonzero impedance, fluctuations in the gate potential will arise from fluctuations in the number of charge carriers in the gate region. These gate potential fluctuations cause additional source-drain current fluctuations (called here amplified backaction noise).

Using restrictions on the noise performance of (phase insensitive) transistor amplifiers, we present a procedure for an experimental identification of the region in parameter space where quantum-limited noise performance is allowed (if such a region exists). Constructed for practical purposes, this procedure makes use of only the knowledge of quantities which are directly measurable. Neither a knowledge of the Hamiltonian of the signal source nor that of the transistor is required. As an example, we show how this procedure can achieve the Heisenberg limit in certain resonant barrier transistors.

We begin by introducing the restrictions on the noise performance of transistor amplifiers. Consider a signal carried by a current \(I_{in}\) which is flowing out of a source having a differential conductance \(g_{\ell}\) and which enters the amplifier input port. The resulting amplified signal \(I_{out}\) is delivered to a load resistor, having a differential conductance \(g_s\), connected to the amplifier output port. We shall consider an amplifier which is impedance-matched to the load; i.e., it has an impedance \(g_s^{-1}\) at its output port. The constraints presented below hold for this case. However, the noise minimization procedure which is derived from them holds also in the general case of impedance mismatch. If \(I_{out}(t)\) is proportional to \(I_{in}(t)\), the amplifier is called linear (and phase insensitive). One can then define the power gain \(G^2\) of the amplifier by the input-output relation

\[ I_{out}(t) = G \frac{g_{\ell}}{g_s} I_{in}(t) + I_{N}(t), \]

where

\[ I_{out}(t) = e^{iH_{out}t} I_{out}(0) e^{-iH_{out}t}, \quad I_{in}(t) = e^{iH_{in}t} I_{in}(0) e^{-iH_{in}t}, \quad H_{tot} = H_a + H_s + \gamma H_{a,s} \]

is the total Hamiltonian, \(H_a\) is the Hamiltonian of the signal source, \(H_s\) is that of the amplifier, and \(\gamma H_{a,s}\) is that of the interaction between them. \(\gamma\) is a small dimensionless coupling constant. \(I_{N}\) is called the noise current operator and is a function of operators related to the amplifier degrees of freedom and, therefore, commutes with \(I_{in}\): \([I_{N}(t), I_{in}(t)] = 0\). \(I_{N}\) is called “noise” because, according to Eq. (1), if the source is prepared in an eigenstate of \(I_{in}\) with an eigenvalue \(i_{in}\), a single measurement of \(I_{out}\) would yield the value \(G \sqrt{g_{\ell} / g_s} I_{in} + \Delta I_{N}\), where \(\Delta I_{N} \equiv \langle I_{N}^2 \rangle - \langle I_{N} \rangle^2\) (the average is taken with respect to the amplifier state).
If the signal source and the amplifier are initially prepared in stationary states and if, after switching on the coupling, they remain in stationary states, although modified ones, and if the amplifier remains approximately impedance-matched, then [6] $\Delta I_{\text{amp}}^2 \approx (G^2 - 1) \times \left(\frac{\hbar \omega_0}{2} g_i \Delta \nu\right)$, where $\Delta \nu = \Delta \omega / (2 \pi)$ is the detection bandwidth and $\Delta \omega$ is a narrow spread of frequencies around the center frequency $\omega_0$ of the band in which the detection is performed. This inequality is a constraint on the total amplifier noise. Defining the idling-noise current by $I_0 \equiv I_N(\gamma = 0)$ and the amplified backaction noise current by $I_n \equiv I_N(\gamma) - I_0$ and assuming these two contributions have zero mean (for $\omega_0 \neq 0$) and are uncorrelated, $\langle I_0 I_n \rangle = 0$, one has $\Delta I_{\text{amp}}^2 = \Delta I_0^2 + \Delta I_n^2$, so that the above inequality restricts the sum of the two types of noise. Assuming that $I_n \sim \gamma^2$, it is shown below that their product is restricted by the condition [7,11]

$$\Delta I_0(t) \Delta I_n(t) \geq G^2 \frac{\hbar \omega_0}{4} g_i \Delta \nu, \quad (2)$$

which implies that the Heisenberg limit for transistor amplifiers with a large gain $G^2 \gg 1$ is achieved if and only if

$$\Delta I_0^2 = \Delta I_n^2 = G^2 \frac{\hbar \omega_0}{4} g_i \Delta \nu. \quad (3)$$

Equation (2) resembles constraints derived for general linear detectors [4,5] or specific ones [8,10]. It differs from these results in that it contains only directly measurable quantities: the noise contributions one would measure at the output, the gain, and the conductance.

Equation (3) has several nontrivial consequences. It shows that the initial idling noise $\Delta I_0^2(t)$ should not be made too small since coupling a device with a vanishing idling noise to a signal will result in the appearance of an amplified backaction noise $\Delta I_n^2(t)$, which will diverge in order to maintain the inequality in Eq. (2). In particular, for ideal operation of the amplifier at a given gain, the amplified backaction noise and the idling noise should be each equal to half of the amplified zero point fluctuations of the amplifier.

Before presenting a way to reach the condition Eq. (3) in practice, we outline the derivation of Eq. (2) [for details, see Ref. [7]]. Applying the nonequilibrium Kubo formula [13–15] to the amplifier and the source, one has

$$\int_{-\infty}^{\infty} dt e^{i\omega t} \langle [I_a(t), I_a(0)] \rangle = 2 \hbar \omega g_a, \quad \alpha = a, s. \quad (4)$$

$g_a = g_i$ is the source-drain differential conductance of the amplifier. $I_a$ is the unperturbed current signal (i.e., the source current in the absence of coupling to the amplifier). $I_0$ is the current that would flow out of the amplifier if the load resistor is replaced by a short [6]. The impedance matching implies that $I_s = 2I_m$ and $I_v = 2I_{\text{out}}$. Denoting $I(\omega) = \langle I_0(\omega) \rangle$ and $I_0(\omega) \equiv \int_{0+}^{\infty} do I(\omega)e^{i\omega t}d\omega$ and using Eqs. (1) and (4) and the fact that $I_m$ and $I_N$ commute, one has

$$\langle [I_N(\omega_0), I_N^{\dagger}(\omega_0)] \rangle = -(G^2 - 1) \frac{\hbar \omega_0}{2} g_i \Delta \omega. \quad (5)$$

Subtracting Eq. (5) written for $\gamma > 0$ from itself written for $\gamma = 0$ and neglecting terms with a higher order than $\gamma^3$, one obtains $\langle \{I_0(\omega_0), I_0^{\dagger}(\omega_0)\} + \text{H.c.} \rangle = -\pi G^2 \hbar \omega_0 g_i \Delta \nu$. Written as an expectation value of a commutator [7] $\langle [I_0(\omega_0), I_0^{\dagger}(\omega_0), ii[I_0(\omega_0) - I_0(\omega_0)]\} &= -i\pi G^2 \hbar \omega_0 g_i \Delta \nu$, this leads to the uncertainty relation Eq. (2).

We now present a noise minimization procedure aimed at obtaining the two equalities in Eq. (3) in devices in which the Heisenberg limit is achievable. This procedure requires certain practical conditions to hold, the main one being that the coupling $\gamma$ between the signal source and the transistor gate can be smoothly controlled over a wide range of values. It is also taken for granted that the source-drain bias voltage $V$ is well controlled. The control of the coupling can be achieved, for example, by a control of the gate capacitance. The procedure involves only the knowledge of measurable quantities—there is no need to calculate in advance the $V$ and $\gamma$ dependence of the noise. The procedure consists of two simple steps which we refer to as noise balancing and gain matching. In the first step, one varies the coupling and the bias voltage until they reach two values $\gamma_1$ and $V_1$, where the two types of noise reach the same value:

$$\Delta I_0^2(V_1, \gamma_1) = \Delta I_0^2(V_1). \quad (6)$$

The functional dependence of the idling noise on $V$ and $\gamma$ differs from that of the amplified backaction noise (e.g., $I_0 \sim \gamma^0$ while $I_n \sim \gamma^2$). Equating the two types of noise should therefore be possible by varying either $\gamma$ or $V$. The variation of both (and of other controllable parameters) is, in general, necessary in order to maintain the linearity of the amplifier. The noise balancing does not imply noise minimization, and the total noise may even increase during this step. In order to describe the step that follows noise balancing, two power gains are defined: The first, the signal power gain $G_0^2(V_1, \gamma_1)$, is determined by a direct gain measurement. The second, the noise power gain $G_n^2(V_1)$, is calculated using the relation:

$$\Delta I_0^2(V_1) = G_0^2(V_1) \frac{\hbar \omega_0}{4} g_i \Delta \nu. \quad (7)$$

$\langle \hbar \omega_0 / 4 g_i \Delta \nu \rangle$ is half the power delivered by the zero point fluctuations of the amplifier to the load. Therefore, $G_0^2$ is the idling noise referred to this power. The second step consists of matching the two gains by varying the bias voltage and the coupling until $G_n^2(V) = G_0^2(V, \gamma)$. This should be done while maintaining the condition

$$\gamma G(\gamma, V) = \text{const.} \quad (8)$$

If $G \sim \gamma V$ (as is often the case), Eq. (8) means that the gain matching is performed while keeping the product of $\gamma^2$ and the voltage constant: $\gamma^2 V = \gamma V_1$. Equation (8) ensures that the gain matching is performed while keeping the
idling noise and amplified backaction noise balanced as in Eq. (6), and, therefore, the condition given by Eq. (3) (and thus also the Heisenberg limit) is achieved.

It remains to explain why the condition Eq. (8) ensures that the two types of noise remain equal while the gains are matched. For this, we consider the origin of the amplified backaction noise. Because of the linear coupling, a current fluctuation of order $\Delta I_0$ in the transistor induces a fluctuation of order $\gamma \Delta I_0$ in the signal source. This fluctuation is amplified and contributes a noise power $\sim \gamma^2 G^2 \Delta I_0^2$ to the output signal. This extra noise is the amplified backaction $\Delta I_0^2$. Thus,

$$\frac{\Delta I_0^2}{\Delta I_0^2} \sim \gamma^2 G^2,$$

which means that the ratio of the idling noise and amplified backaction noise remains constant if $\gamma^2 G^2$ does.

A typical example is where the idling noise is a shot noise; i.e., it results from the partitioning of charges between the two sides of a tunneling barrier in the source-drain current path. The transfer of a fraction of this noise into the signal source stems from transitions enabled by the appearance of new scattering channels in the presence of the signal source where passing electrons transfer a quantum of $h\omega_0$ to the signal source. The total contribution of these processes is proportional to the number of electrons in the transistor which can participate in such transitions.

At zero temperature, and if $h\omega_0 \ll eV$, all electrons in the nonequilibrium energy window created by the voltage $V$ may undergo such transitions, and, therefore, the number of these transitions is $\sim V$. Thus, the power emitted into the source is $\sim \gamma^2 V$. After amplification, the contribution of these additional fluctuations in the signal current, is $\Delta I_0^2 \sim \gamma^2 V G^2$. On the other hand, the (low frequency) shot-noise power is [16] $\Delta I_0^2$ shot noise $\sim V$. These two estimates confirm Eq. (9).

We now illustrate our results for the specific case of a signal amplified by a resonant barrier transistor coupled capacitatively to a continuum of LC resonators (quantum harmonic oscillators) that models a resistive signal source. The model is similar in many features to those analyzed in Refs. [8,9,17]. The total Hamiltonian is

$$H_{\text{tot}} = \sum_{i=1,2} \int_0^\infty d\epsilon e \varepsilon_b \left( \epsilon_b \right) + h\omega_0 A^\dagger A$$

$$+ \sum_{i=1,2} A^\dagger A\varepsilon_b \left( \epsilon_b \right) (e \varepsilon_a \left( \epsilon_a \right) - A^\dagger b_i \left( \epsilon_i \right))$$

$$+ \frac{A^\dagger A\varepsilon_b \left( \epsilon_b \right)}{C_g} + \int_B d\omega h\omega_a A^\dagger (\omega) a(\omega),$$

where $\tilde{Q}_s = \Delta Q(\omega_0) \int_B d\omega (1/\sqrt{\omega_0}) [a(\omega) + a^\dagger (\omega)]$ is the total charge on the capacitors in the LC oscillators and where $B = [\omega_0 - \Delta \omega/2, \omega_0 + \Delta \omega/2]$. The $b_i$'s, $A$'s, and $a(\omega)$'s satisfy, respectively, continuous fermionic, discrete fermionic, and continuous bosonic commutation relations. $b_i$ annihilates an electron in bath $i = 1, 2$. $A$ annihilates an electron in the resonance level which is located at energy $h\omega_0$, $k(\epsilon)$ is the tunneling amplitude between the baths and the resonance level, $\epsilon$ is the single-electron energy, $k^2(\epsilon)$ is the resonance width, is taken to be wider than $eV$ so that the second derivative of the transmission with respect to $\epsilon$ (but not the first) can be neglected. It is also assumed that $k^2(\epsilon)$ is small compared to $h\omega$ and the Fermi energy. $C_g$ is the gate capacitance of the amplifier, and $\Delta Q(\omega_0)$ is the typical charge fluctuation in one of the oscillators in its ground state, $\Delta Q = \sqrt{h\omega_0 C/2}$, where $C$ is the capacitance in each one of the LC circuits. Denoting the coupling constant by $\gamma = e\Delta Q/(C_g k^2)$ and assuming $\gamma \ll 1$, the coupling term in $H_{\text{tot}}$ can be written as $A^\dagger A\varepsilon_b \left( \epsilon_b \right) / C_g = yk^2 A^\dagger \tilde{Q}_s / \Delta Q$, which plays the role of $\gamma H_{\text{g.s.}}$ above. The principle of operation of this transistor amplifier is the following: The signal modulates the position of the resonant level and, hence, the transmission. In the classical picture, this modulates the output current. In the quantum picture, this creates inelastic components for the transmitted electrons which lead to a structure (proportional to the square of a large bias voltage) mirroring the signal power spectrum in the output current power spectrum.

The transistor is taken to be in a zero-temperature stationary state with baths 1 and 2 having chemical potentials $\mu + eV$ and $\mu$ and, thus, occupation numbers $n_1(\epsilon) = \Theta(\epsilon) \Theta(\mu + eV - \epsilon)$ and $n_2(\epsilon) = \Theta(\epsilon) \Theta(\mu - \epsilon)$, respectively. The transistor current operator is defined by the rate of change in the charge of the two baths:

$$I_\omega(t) = \frac{\hbar}{2} \tilde{Q}_s(t) - \tilde{Q}_s(t),$$

where $\tilde{Q}_s(t) = \int_0^\infty d\epsilon e \varepsilon_b \left( \epsilon_b \right) \varepsilon_b \left( \epsilon_b \right) t$, $\varepsilon_b \left( \epsilon_b \right)$ is the total charge in bath $i$. Solving the Heisenberg equations of motion to second order in $\gamma$, we find (recall: $I_{\text{in}} = \frac{1}{2} I_0$)

$$I_{\text{out}}(t) = I_0(t) + G \sqrt{\frac{g_{\text{in}}}{g_\omega}} I_{\text{in}}(t) + I_{\text{n}}(t) + O(\gamma^3),$$

where $I_{\text{in}} = \frac{1}{2} \omega_0 Q_s(t)$, and

$$I_0(t) = \frac{e\hbar}{4\pi} \int_B d\omega e^{-i\omega t} \int_0^\infty d\psi' \left[ t^*(\omega') b_+ (\omega') b_+ (\omega' + \omega) + t(\omega') b_+ (\omega' - \omega) b_+ (\omega') \right]$$

is the $\gamma = 0$ current, $\pm B = [\omega_0 - \Delta \omega, \omega_0 + \Delta \omega] \cup [\omega_0 - \Delta \omega, \omega_0 + \Delta \omega]$, $b_+ = \frac{1}{\sqrt{2}} (b_1 + b_2)$, $b_- = \frac{1}{\sqrt{2}} (b_1 - b_2)$,

$$G = \gamma eV / \hbar \omega_0 T \sqrt{2(1 - T)},$$

$t(\omega) = -k^2/[ih(\omega - \omega_0) + k^2]$ is the transmission amplitude at energy $h\omega$, $T = |t|^2$. $I_{\text{out}}(t)$ is the amplified backaction noise current, the explicit expression for which will not be given here. Note that Eq. (12) is an operator input-output relation and, therefore, enables one to calculate expectation values of any function of $I_{\text{in}}, g_\omega = T e^2 / 2\pi\hbar$.
and \( \tilde{g}_s \equiv \pi \Delta Q^2 / h \). \( \tilde{g}_s \) is the differential linear response of the “current” \( I_e = \omega_0 Q_s \). Comparing Eqs. (1) and (12), one sees that actually the device performs linear amplification of \( I_e \) instead of \( I_e = Q_s \). This is a consequence of the capacitive coupling \( H_{s,e} = eA^1AQ_s / C_s \). However, Eq. (4) is valid also for \( \tilde{I}_e \) and so are all the above results—the only modification one needs to apply is the replacement of \( \tilde{g}_s \) by \( \tilde{g}_s \) as done in Eq. (12).

Equation (14) implies that a large gain \( G^2 \gg 1 \) requires a stronger assumption than \( eV \gg \hbar \omega_0 \), namely, \( eV \gg \hbar \omega_0 \gamma^{-1} \). We also note that, when solving the Heisenberg equations, the coefficient before \( Q_s \) in Eq. (12) turns out to be an operator \( \tilde{G} \) [Eq. (2), with \( G \rightarrow \langle \tilde{G} \rangle \) is still valid in this case]. However, for a narrow bandwidth signal \( \hbar \Delta \omega \ll eV \), the quantum fluctuations of this operator are negligible \( \Delta \tilde{G}^2 \ll \langle \tilde{G} \rangle^2 \equiv G^2 \). This allows us to replace it by its expectation value.

From Eqs. (12)–(14), one obtains the idling noise:

\[
\Delta I^2_0 = T(1 - T) \frac{e^3 V}{4 \pi h} \Delta \nu. \tag{15}
\]

A lengthier calculation yields the amplified backaction noise

\[
\Delta I^2_n = \frac{\gamma^2}{4} T^2 (1 - T) \frac{e^3 V}{\hbar \omega_0} \frac{e^3 V}{\pi h} \Delta \nu. \tag{16}
\]

One also finds that these noise sources are indeed uncorrelated \( \langle I_n I_0 \rangle = 0 \). Equations (14)–(16) yield

\[
\Delta I^2_n(\tau) \sim \frac{1}{2} \gamma^2 \hbar \omega_0 \Delta \nu. \tag{17}
\]

Equations (15)–(17) demonstrate how an amplifier satisfying the constraint Eq. (2) as an equality may still not be operating at the Heisenberg limit. To achieve this limit, the noise balancing should be performed. Equating \( \Delta I^2_n = \Delta I^2_0 \) yields the condition

\[
\frac{\gamma^2}{4} \frac{e^3 V}{\hbar \omega_0} T^2 = 1. \tag{18}
\]

By Eqs. (3), (7), and (14)–(16), any pair of \( \gamma \) and \( V \) satisfying Eq. (18) results in performance at the Heisenberg limit (i.e., here \( G_N = G \)). Equations (14) and (18) imply that \( \gamma G = \text{const} \), confirming Eq. (9). To identify all possible values for the gain at the Heisenberg limit \( G^H \), we insert Eq. (18) into Eq. (14) and find \( G^H = \sqrt{2(1 - T)} / (\gamma T) \). One should recall the assumption that the second derivative of the transmission vanishes, which is strictly true only when \( T = 3/4 \). Thus,

\[
G^H = \frac{2\sqrt{2}}{3} \frac{1}{\gamma}. \tag{19}
\]

To summarize, we presented a practical procedure for finding the region in parameter space where transistor amplifiers achieve the optimum noise performance allowed by quantum mechanics for linear phase insensitive amplifiers. The procedure should be experimentally feasible for linear devices for which such a parameter region exists even if the precise Hamiltonian of the device is unknown. We then verified the validity of this procedure in the case of a resonant barrier transistor amplifier coupled to a resistive signal source modeled as a continuum of LC resonators.

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[10] A. A. Clerk, Phys. Rev. B 70, 245306 (2004). Equation (14) in that work can be shown to be consistent with Eq. (2) here when combining the assumptions in both works.
[11] This constraint is valid for a transistor which is impedance-matched to a load resistor connected to its output port, which is what we shall assume throughout this work. Although this assumption affects the absolute value of the noise, it does not affect the signal-to-noise ratio. If, e.g., the load resistor is a short (and is therefore no longer impedance-matched to the transistor), both the noise and the signal powers are increased by a factor of 4 with respect to their values in the impedance-matched setup, while the signal-to-noise ratio is unchanged.
[12] \( g_s(\omega) \) is the linear response to a small ac field applied to a system in a stationary state. If this state is an equilibrium, \( g_s(\omega) \) is the ordinary conductance.
[15] See also Ref. [6], and references therein.