Superfluid pairing between fermions with unequal masses

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We consider a superfluid state in a two-component gas of fermionic atoms with equal densities and unequal masses in the BCS limit. We develop a perturbation theory along the lines proposed by Gorkov and Melik-Barkhudarov and find that for a large difference in the masses of heavy (M) and light (m) atoms one has to take into account both the second-order and third-order contributions. The result for the critical temperature and order parameter is then quite different from the prediction of the simple BCS approach. Moreover, the small parameter of the theory turns out to be $(p_F|a|/\hbar)\sqrt{M/m} \ll 1$, where p_F is the Fermi momentum and a the scattering length. Thus, for a large mass ratio M/m the conventional perturbation theory requires significantly smaller Fermi momenta (densities) or scattering lengths than in the case of $M \sim m$, where the small parameter is $p_F|a|/\hbar \ll 1$. We show that three-body scattering resonances appearing at a large mass ratio due to the presence of three-body bound Efimov states do not influence the result, which in this sense becomes universal.

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I. INTRODUCTION

Superfluid pairing in a two-component gas of fermions is a well-known problem [1] lying in the background of extensive studies in condensed matter and nuclear physics [2–4]. Recently, this problem was actively investigated in cold gases of fermionic atoms (see [5] for a review). Experimental efforts were focused on ⁶Li or ⁴⁰K atoms in two different internal (hyperfine) states, where one can use Feshbach resonances for switching the sign and tuning the absolute value of the interspecies interaction (scattering length a), which at resonance changes from $-\infty$ to $+\infty$. In this respect, one encounters the problem of BCS–Bose-Einstein-condensation (BEC) crossover discussed earlier in the context of superconductivity [6–10] and for superfluidity of two-dimensional ³He films [11,12]. On the negative side of the resonance ($a < 0$), one should have the Bardeen-Cooper-Schrieffer (BCS) superfluid pairing at sufficiently low temperatures, and on the positive side ($a > 0$) one expects Bose-Einstein condensation of diatomic molecules formed by atoms of different components. Remarkable achievements of cold-atom physics in the last years include the observation of superfluid behavior through vortex formation in the strongly interacting regime ($n|a|^3 \gtrsim 1$, where n is the gas density) [13], and the formation and Bose-Einstein condensation of long-lived weakly bound diatomic molecules at $a > 0$ [14]. Ongoing experiments with atomic Fermi gases have reached temperatures in the nanokelvin regime, where at achieved densities one has $T \sim 0.1 E_F$, with E_F being the Fermi energy. For $a < 0$ the experiments are now approaching superfluidity in the BCS limit where $n|a|^3 \ll 1$.

Currently, a new generation of experiments is being setup. In particular, it is dealing with mixtures of different fermionic atoms or mixtures of fermions and bosons. The main goal is to reveal the influence of the mass difference on superfluid properties and to search for types of superfluid pairing. The first experiments demonstrating a possibility of using Feshbach resonances and creating collisionally stable mixtures of ⁴⁰K with ⁶Li and/or with ⁸⁷Rb, and ⁶Li with ²³Na have already been performed [15–20]. Recent theoretical literature on mixtures of different fermionic atoms contains a discussion of the BCS limit [21–23], the limit of molecular BEC [24], BCS-BEC crossover [25–27], and the strongly interacting regime [28].

In this paper we consider a two-component mixture of fermionic atoms with different masses and attractive inter-component interaction in the BCS limit. It is assumed that the densities of the two species are equal which means that there is no mismatch between their Fermi surfaces, leading to the usual BCS type of superfluid pairing. Other kinds of pairing that can occur and compete with BCS, especially for unequal densities, will be discussed elsewhere [29]. Here, we generalize the perturbation treatment of the gap equation, introduced by Gorkov and Melik-Barkhudarov [30] for equal masses of fermions belonging to different components. This approach takes into account the interaction between the atoms in a Cooper pair due to the polarization of the medium and allows one to correctly determine the dependence of the zero-temperature gap Δ_0 and superfluid transition temperature T_c on the masses of heavy (M) and light (m) fermionic atoms. As we shall see below, already the second order of the perturbation, the so-called Gorkov-Melik-Barkhudarov contribution, leads to a very different dependence of the preexponential factor in the expressions for Δ_0 and T_c on the mass ratio M/m, compared to the prediction of the simple BCS theory.

For a large mass ratio $M/m \gg 1$, we include higher order contributions and show that the actual small parameter of the theory is $(p_F|a|/\hbar)\sqrt{M/m} \ll 1$ (p_F is the Fermi momentum), not simply $p_F|a|/\hbar \ll 1$ as in the case of $M \sim m$. We give a physical interpretation of this fact and calculate effective masses of heavy and light fermions.

Large mass ratios M/m are realized in electron-ion plasmas, where the heavy ion component is usually considered as nondegenerate [31]. The electron-proton pairing in the hydrogen plasma, assuming quantum degeneracy for both electrons and protons, was discussed by Moulopoulos and Ashcroft [32]. They found that at low temperatures the Coulomb electron-proton attraction leads to the appearance of a (momentum-dependent) gap which for sufficiently high densities is comparable with the Coulomb interaction at the mean interparticle separation. Note that this problem is quite different from ours where the attractive interaction between heavy and light fermions is short ranged.

Before proceeding with our analysis we make two important remarks. First of all, if the masses of heavy and light fermionic atoms are very different from each other and the mass ratio exceeds a critical value, $M/m > 13.6$, then two heavy and one light fermion can form three-body weakly bound states. The appearance of these states, predicted by Efimov [33], can be easily understood in the Born-Oppenheimer picture [40]. If we fix the two heavy atoms at a relative distance $R < |a|$, a localized state for the light atom appears due to the presence of the heavy pair, which in turn mediates an attractive interaction $\sim -\hbar^2/mR^2$ between the heavy atoms (see, e.g., Ref. [24] and references therein). For a large mass ratio, $M/m > 13.6$, this mediated attraction overcomes the kinetic energy of the relative motion of the heavy atoms and one has the well-known phenomenon of "fall into center" [34]. The energy of this state is bounded from below only due to short-range repulsion. The corresponding wave function of the relative motion of heavy atoms acquires a large number of nodes thus showing the presence of many bound states. This makes the three-body problem nonuniversal in the sense that aside from the two-body scattering length a, the description of this problem requires one more parameter—the so-called three-body parameter coming from short-range physics. Also, the presence of weakly bound Efimov states introduces a resonant character to the three-body scattering problem. This is especially important for the Gorkov-Melikh-Barkhudarov contribution as it is actually dealing with processes involving three particles. We, however, have found that the three-body resonances are rather narrow and their contribution is not important. This makes the Gorkov-Melikh-Barkhudarov approach universal at any mass ratio M/m.

Our second remark is related to analogies between BCS pairing of unequal-mass particles in cold-atom and high energy physics. We wish to emphasize that there are strong physical differences between the pairing of particles of different masses in relativistic and in nonrelativistic systems such as cold atoms. The problem arises in relativistic systems in the study of hadronic matter [35]. It is thought that, at the high densities achieved in neutron stars, quarks become deconfined and the different types of them (e.g., up, down, and strange quarks) will tend to form Cooper pairs with each other. These different types of quarks have different masses. This relativistic limit has been investigated by Kundu and Rajagopal [36] and is characterized by the Fermi momentum being larger than the bare mass: $p_F \gg mc$. Linearizing the momentum near the Fermi surface $p = p_F + \delta p$, $(\delta p/p_F \ll 1)$, we can expand the free particle energy to the lowest nonvanishing order in $\delta p/p_F$ and mc/p_F:

$$E = \sqrt{p^2 c^2 + m^2 c^4} \simeq p_F c \left(1 + \frac{\delta p}{p_F} + \frac{m^2 c^2}{2 p_F^2} \right).$$

We see that, as far as the kinetic energy is concerned, a change in mass will amount to a shift in the chemical potential of the species which is proportional to m^2 and depends inversely on p_F. Therefore, pairing between particles with different masses in the relativistic limit is equivalent to studying the problem of pairing of equal mass particles in the presence of a difference between the chemical potentials of the two species. In the nonrelativistic limit ($p_F \ll mc$) the situation is different,

$$E \simeq mc^2 + \frac{p^2}{2m}$$

and so the mass change cannot be incorporated into the chemical potential, requiring a very different analysis which we carry out here.

The paper is organized as follows. In Sec. II we present general equations, and in Secs. III and IV we calculate the critical temperature T_c. Section V is dedicated to the discussion of the small parameter of the theory, and in Sec. VI we discuss the order parameter and excitation spectrum. In Sec. VII we analyze the three-body resonances at a large mass ratio M/m and show that they do not change the result of the Gorkov-Melik-Barkhudarov approach. In Sec. VIII we conclude.

II. GENERAL EQUATIONS

We consider a uniform gas composed of heavy and light fermionic atoms with masses M and m, respectively. Both heavy and light atoms are in a single hyperfine state, and considering low temperatures we omit heavy-heavy and light-light interactions. The interaction of heavy atoms with light ones is assumed to be attractive and characterized by a negative s-wave scattering length $a < 0$. The Hamiltonian of the system has the form

$$H = \int d\mathbf{r} \left[\sum_{i=1,2} \hat{\psi}_i^+(\mathbf{r}) \left(-\frac{\hbar^2}{2m_i} \nabla^2 - \mu_i \right) \hat{\psi}_i(\mathbf{r}) \right.$$
$$\left. + g \hat{\psi}_1^+(\mathbf{r}) \hat{\psi}_1(\mathbf{r}) \hat{\psi}_2^+(\mathbf{r}) \hat{\psi}_2(\mathbf{r}) \right], \qquad (1)$$

where $\hat{\psi}_i(\mathbf{r})$ and $\hat{\psi}_i^+$ are the field operators of fermionic atoms labeled by indices $i=1$ (heavy) and $i=2$ (light), μ_i is the corresponding chemical potential, and $g = 2\pi\hbar^2 a/m_r$ is the coupling constant, with $m_r = Mm/(M+m)$ being the reduced mass. Since the densities of the two species are equal, $n_1 = n_2 = n$, they have the same Fermi momentum $p_F = \hbar(6\pi^2 n)^{1/3}$, and, hence, $\mu_1 = p_F^2/2M$ and $\mu_2 = p_F^2/2m$. Finally, we require that the system be in the weakly interacting regime, which requires the inequality $p_F|a|/\hbar \ll 1$.

We now consider the usual BCS scheme where a heavy atom with momentum $\mathbf{p}$ is paired to a light one having momentum $-\mathbf{p}$. This leads to the gap equation
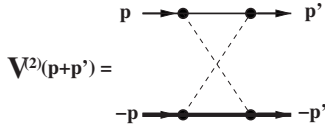
FIG. 1. The leading many-body contribution (second order, or Gorkov-Melik-Barkhudarov correction) to the effective interaction between heavy (thick line) and light (thin line) fermions. The dashed line corresponds to the coupling constant g.

$$\Delta(\mathbf{p}) = -\int \frac{d\mathbf{p}'}{(2\pi\hbar)^3} V_{\text{eff}}(\mathbf{p},\mathbf{p}')$$
$$\times \frac{1 - f(E_+(\mathbf{p}')) - f(E_-(\mathbf{p}'))}{E_+(\mathbf{p}') + E_-(\mathbf{p}')}\Delta(\mathbf{p}'), \qquad (2)$$

where $f(E) = [\exp(E/T)+1]^{-1}$ is the Fermi-Dirac distribution function, and we assume that the order parameter is real. The dispersion relations for the two branches of single-particle excitations are written as

$$E_\pm(\mathbf{p}) = \pm\left(\frac{\xi_1(\mathbf{p}) - \xi_2(\mathbf{p})}{2}\right) + \sqrt{\left(\frac{\xi_1(\mathbf{p}) + \xi_2(\mathbf{p})}{2}\right)^2 + \Delta^2},$$
$$(3)$$

and the quantities $\xi_{1,2}$ are given by $\xi_1(\mathbf{p}) = (p^2 - p_F^2)/2M$ and $\xi_2(\mathbf{p}) = (p^2 - p_F^2)/2m$. The function $V_{\text{eff}}(\mathbf{p},\mathbf{p}') = g + \delta V(\mathbf{p},\mathbf{p}')$ is an effective interaction between particles in the medium, where the quantity $\delta V(\mathbf{p},\mathbf{p}')$ originates from many-body effects and is a correction to the bare interparticle interaction g. The leading correction is second order in g and the corresponding diagram is shown in Fig. 1.

The integral in Eq. (2) diverges at large momenta due to the first term in V_{eff}. This divergency can be eliminated by expressing the bare interaction g in terms of the scattering length a [1,30,37]. If we confine ourselves to the second order in perturbation theory with respect to g, then the renormalized gap equation reads

$$\Delta(\mathbf{p}) = -\frac{2\pi\hbar^2 a}{m_r} \int \frac{d\mathbf{p}'}{(2\pi\hbar)^3}$$
$$\times \left[\frac{1 - f(E_+(\mathbf{p}')) - f(E_-(\mathbf{p}'))}{E_+(\mathbf{p}') + E_-(\mathbf{p}')} - \frac{2m_r}{p'^2}\right]\Delta(\mathbf{p}')$$
$$-\int \frac{d\mathbf{p}'}{(2\pi\hbar)^3} \delta V(\mathbf{p},\mathbf{p}')\frac{1 - f(E_+(\mathbf{p}')) - f(E_-(\mathbf{p}'))}{E_+(\mathbf{p}') + E_-(\mathbf{p}')}\Delta(\mathbf{p}').$$
$$(4)$$

The convergence of the integral over p' in the first term of the right-hand side of Eq. (4) is now obvious, while the convergence of the second term is due to the decay of δV at large momenta [see Eq. (11) below]. The gap equation accounting for higher orders in g will be derived and discussed in Sec. IV.

In the limit of $\Delta \to 0$ one can reduce Eq. (4) to a linearized gap equation,

$$\Delta(\mathbf{p}) = -\frac{2\pi\hbar^2 a}{m_r} \int \frac{d\mathbf{p}'}{(2\pi\hbar)^3}$$
$$\times \left[\frac{\tanh[\xi_1(\mathbf{p}')/2T] + \tanh[\xi_2(\mathbf{p}')/2T]}{2[\xi_1(\mathbf{p}') + \xi_2(\mathbf{p}')]} - \frac{2m_r}{p'^2}\right]\Delta(\mathbf{p}')$$
$$-\int \frac{d\mathbf{p}'}{(2\pi\hbar)^3} \delta V(\mathbf{p},\mathbf{p}')$$
$$\times \frac{\tanh[\xi_1(\mathbf{p}')/2T] + \tanh[\xi_2(\mathbf{p}')/2T]}{2[\xi_1(\mathbf{p}') + \xi_2(\mathbf{p}')]}\Delta(\mathbf{p}'). \qquad (5)$$

The critical temperature T_c is determined from Eq. (5) as the highest temperature at which this equation has a nontrivial solution for Δ.

III. CRITICAL TEMPERATURE. BCS AND GORKOV-MELIK-BARKHUDAROV (GM) APPROACHES

The first line of Eq. (5) corresponds to the linearized gap equation in the traditional BCS approach,

$$\Delta(\mathbf{p}) = -\frac{2\pi\hbar^2 a}{m_r} \int \frac{d\mathbf{p}'}{(2\pi\hbar)^3}$$
$$\times \left[\frac{\tanh[\xi_1(\mathbf{p}')/2T] + \tanh[\xi_2(\mathbf{p}')/2T]}{2[\xi_1(\mathbf{p}') + \xi_2(\mathbf{p}')]} - \frac{2m_r}{p'^2}\right]\Delta(\mathbf{p}').$$
$$(6)$$

In this case the order parameter is momentum independent, $\Delta(\mathbf{p}) = \Delta$, and Eq. (6) reduces to the equation for the critical temperature,

$$1 = \frac{\lambda}{2}\left[\ln \frac{8\mu_1 \exp(\gamma - 2)}{\pi T_{BCS}} + \ln \frac{8\mu_2 \exp(\gamma - 2)}{\pi T_{BCS}}\right], \qquad (7)$$

where $\gamma = 0.5772$ is the Euler constant and we introduced a small parameter

$$\lambda = 2p_F|a|/\pi\hbar \ll 1. \qquad (8)$$

Equation (7) is obtained straightforwardly. First, integrating Eq. (6) over the angles one has

$$1 = \frac{\lambda}{2}\int_0^\infty dx\{\tanh[(x^2 - 1)\mu_1/2T_c]$$
$$+ \tanh[(x^2 - 1)\mu_2/2T_c] - 2\}$$
$$+ \frac{\lambda}{2}\int_0^\infty dx \frac{\tanh[(x^2 - 1)\mu_1/2T_c] + \tanh[(x^2 - 1)\mu_2/2T_c]}{x^2 - 1},$$

where $x = p/p_F$. For $\mu_i/T_c \gg 1$ the integrand of the first integral is equal to -4 for $x < 1$, and for $x > 1$ it rapidly drops to 0 in a narrow interval of x, where $|x-1| \lesssim T_c/\mu \ll 1$. The contribution of this interval can be neglected and, therefore, the first term equals -2λ. Then, after integrating the second term by parts, we obtain

$$1 = \lambda \left\{ -2 - \frac{1}{2} \int_0^\infty dx \left[\frac{x\mu_1/T_c}{\cosh^2[(x^2-1)\mu_1/2T_c]} \right. \right.$$
$$\left. \left. + \frac{x\mu_2/T_c}{\cosh^2[(x^2-1)\mu_2/2T_c]} \right] \ln \left| \frac{x-1}{x+1} \right| \right\}.$$

The final integration can easily be performed by using the fact that the integrand is nonzero only in a narrow range of x, where $|x-1| \lesssim T_c/\mu \ll 1$. We can therefore introduce a new variable $y = x - 1$ and extend the limits of integration over y from $-\infty$ to $+\infty$. The equation then reads

$$1 = \lambda \left\{ -2 - \frac{1}{2} \int_{-\infty}^\infty dy \left[\frac{\mu_1/T_c}{\cosh^2(y\mu_1/T_c)} \right. \right.$$
$$\left. \left. + \frac{\mu_2/T_c}{\cosh^2(y\mu_2/T_c)} \right] \ln \frac{|y|}{2} \right\},$$

and performing the integration one arrives at Eq. (7). This equation gives the critical BCS temperature (cf. [22])

$$T_{BCS} = \frac{8}{\pi} \exp(\gamma - 2) \sqrt{\mu_1 \mu_2} \exp\left(-\frac{1}{\lambda}\right). \tag{9}$$

However, the linearized BCS gap equation (6) can only be used for the calculation of the leading contribution to the critical temperature, corresponding to the term $\sim \lambda^{-1}$ in the exponent of Eq. (9). Therefore, only the exponent in this equation is correct. As was shown by Gorkov and Melik-Barkhudarov [30], the preexponential factor in Eq. (9) is determined by next-to-leading order terms, which depend on many-body effects in the interparticle interaction. These are the interactions between particles in a many-body system through the polarization of the medium—virtual creation of particle-hole pairs.

The importance of the many-body effects for the preexponential factor can be understood as follows. After performing the integration over momenta, the gap equation (5) can be qualitatively written as

$$1 = \nu_F V_{\text{eff}} \left[\ln \frac{\mu}{T_c} + C \right], \tag{10}$$

where $\nu_F = m_r p_F / \pi^2 \hbar^3$. In this formula, the large logarithm $\ln \mu/T_c$ comes from the integration over momenta near the Fermi surface, whereas the momenta far from the Fermi surface contribute to the constant C which is of the order of unity. We then write $\nu_F V_{\text{eff}} = -\lambda + a\lambda^2$, where the first term is the direct interparticle interaction and we keep only the

second-order term in the many-body part δV of the effective interaction. It is now easy to see that $\lambda \ln \mu/T_c \sim 1$ and, therefore, the terms $a\lambda^2 \ln \mu/T_c$ and λC are of the same order of magnitude. As a result, both terms have to be taken into account for the calculation of the preexponential factor. Also, note that the contribution of the many-body part of the interparticle interaction comes from momenta near the Fermi surface, which are responsible for the large logarithm $\ln \mu/T_c \sim \lambda^{-1}$. Hence, only the values of δV at the Fermi surface are important.

We now calculate the contribution of the many-body effects to the preexponential factor for the critical temperature. They are usually called Gorkov-Melik-Barkhudarov (GM) corrections. As it was argued above, in the weak coupling limit the most important contributions to the effective interaction are second order in g (the role of high-order terms will be discussed later). In the considered case of a two-component Fermi gas with an s-wave interaction, there is only the contribution shown in Fig. 1, and the corresponding analytical expression reads

$$\delta V(\mathbf{p}, \mathbf{p}') = -g^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{f[\xi_1(\mathbf{k}+\mathbf{q}/2)] - f[\xi_2(\mathbf{k}-\mathbf{q}/2)]}{\xi_1(\mathbf{k}+\mathbf{q}/2) - \xi_2(\mathbf{k}-\mathbf{q}/2)}, \tag{11}$$

where $\mathbf{q} = \mathbf{p} + \mathbf{p}'$. In obtaining this expression we used the zero-temperature distribution function $f[\xi_{1,2}(p)] = \theta(-\xi_{1,2}(p))$, with $\theta(x)$ being the step function. This is legitimate because the finite temperature corrections are proportional to the ratio of the critical temperature to the chemical potential and, therefore, are exponentially small. As can be seen from Eq. (11), the effective interaction $\delta V(\mathbf{p}, \mathbf{p}')$ changes on the momentum scale $p \sim p' \sim p_F$.

We now solve Eq. (5). In this equation, the momentum dependence of the order parameter originates only from the momentum dependence of the many-body contribution to the interparticle interaction and, therefore, contains an extra power of the small parameter λ. As a result, this dependence can be ignored in the first integral on the right-hand side of Eq. (5), and we can simply replace there the order parameter $\Delta(p')$ by its value on the Fermi surface $\Delta(p_F)$. This does not affect the convergence of the integral at large momenta and, hence, changes only the constant C in Eq. (10). The corresponding modification, however, is proportional to the small parameter λ and can be neglected. In the second integral on the right-hand side of Eq. (5), as we have discussed earlier, only momenta p' near the Fermi surface ($p' \approx p_F$) are important, and we can also put $p' = p_F$ in $\Delta(p')$ and $\delta V(\mathbf{p}, \mathbf{p}')$. The gap equation then reads

$$\Delta(\mathbf{p}) = -\frac{2\pi\hbar^2 a}{m_r} \int \frac{d\mathbf{p}'}{(2\pi\hbar)^3} \left[\frac{\tanh[\xi_1(\mathbf{p}')/2T_c] + \tanh[\xi_2(\mathbf{p}')/2T_c]}{2[\xi_1(\mathbf{p}') + \xi_2(\mathbf{p}')]} - \frac{2m_r}{p'^2} \right] \Delta(\mathbf{n}' p_F)$$
$$- \int_{p < \Lambda p_F} \frac{d\mathbf{p}'}{(2\pi\hbar)^3} \delta V(\mathbf{p}, \mathbf{n}' p_F) \frac{\tanh[\xi_1(\mathbf{p}')/2T_c] + \tanh[\xi_2(\mathbf{p}')/2T_c]}{2[\xi_1(\mathbf{p}') + \xi_2(\mathbf{p}')]} \Delta(\mathbf{n}' p_F), \tag{12}$$

where $\mathbf{n}'$ is the unit vector in the direction of $\mathbf{p}'$ and we introduced an upper cutoff Λp_F with $\Lambda \sim 1$, for the purpose of convergence at large momenta. The exact value of Λ is not important because, as we mentioned above, the contribution of

large momenta to this integral has to be neglected. To derive an equation for the critical temperature, we consider Eq. (12) for $\mathbf{p}=\mathbf{n}p_F$ and average it over the directions of $\mathbf{n}$. Taking into account that the order parameter for the *s*-wave pairing can only depend on the absolute value of the momentum, we obtain

$$1 = -\frac{2\pi\hbar^2 a}{m_r}\int\frac{d\mathbf{p}'}{(2\pi\hbar)^3}\left[\frac{\tanh[\xi_1(\mathbf{p}')/2T_c]+\tanh[\xi_2(\mathbf{p}')/2T_c]}{2[\xi_1(\mathbf{p}')+\xi_2(\mathbf{p}')]}-\frac{2m_r}{p'^2}\right]-\overline{\delta V}\int_{p<\Lambda p_F}\frac{d\mathbf{p}'}{(2\pi\hbar)^3}\frac{\tanh[\xi_1(\mathbf{p}')/2T_c]+\tanh[\xi_2(\mathbf{p}')/2T_c]}{2[\xi_1(\mathbf{p}')+\xi_2(\mathbf{p}')]}$$

$$=\lambda\ln\frac{8\sqrt{\mu_1\mu_2}\exp(\gamma-2)}{\pi T_c}-\nu_F\overline{\delta V}\ln\frac{8\sqrt{\mu_1\mu_2}\exp(\gamma+\Lambda-2)}{\pi T_c}\approx(\lambda-\nu_F\overline{\delta V})\ln\frac{8\sqrt{\mu_1\mu_2}\exp(\gamma-2)}{\pi T_c},\tag{13}$$

where

$$\overline{\delta V}=\int\frac{d\mathbf{n}}{4\pi}\int\frac{d\mathbf{n}'}{4\pi}\delta V(\mathbf{n}p_F,\mathbf{n}'p_F)\tag{14}$$

is the *s*-wave component of the many-body interaction. Using Eq. (11) and integrating over the angles in Eq. (14), we obtain

$$\overline{\delta V}=\nu_F g^2\frac{1+\ln 4}{3}[f(\kappa)+f(\kappa^{-1})],\tag{15}$$

where $\kappa=M/m$ and the function $f(\kappa)$ is given by

$$f(\kappa)=-\frac{3(1+\kappa)}{4(1+\ln 4)}\int_0^1 dq\int_0^1 pdp\ln\left|\right.$$
$$\times\frac{(p^2-1)(\kappa-1)+4q(p-q)}{(p^2-1)(\kappa-1)-4q(p+q)}\left|\right..\tag{16}$$

A straightforward lengthy integration of Eq. (16) yields

$$f(\kappa)=-\frac{3}{4(1+\ln 4)}(1+\kappa)\left[-\frac{\kappa+1}{3\kappa}\ln(2)+\frac{\kappa-1}{3\kappa}\ln|\kappa-1|\right.$$
$$\left.+\frac{4}{3(\kappa-1)}-\left(\frac{(\kappa+3)^2}{6(\kappa-1)^2}+\frac{\kappa+2}{6\kappa}\right)\ln\frac{\kappa+1}{2}\right].\tag{17}$$

From Eqs. (13) and (15) we obtain the following expression for the critical temperature

$$T_{GM}=\frac{8}{\pi}e^{\gamma-2}\sqrt{\mu_1\mu_2}\exp[-(\lambda-\nu_F\overline{\delta V})^{-1}]$$

$$\approx T_{BCS}\exp(-\nu_F\overline{\delta V}/\lambda^2)=\frac{e^\gamma}{\pi}\left(\frac{2}{e}\right)^{7/3}$$

$$\times\exp\left\{-\frac{1+\ln 4}{3}[f(\kappa)+f(\kappa^{-1})-1]\right\}$$

$$\times\sqrt{\mu_1\mu_2}\exp\left(-\frac{1}{\lambda}\right).\tag{18}$$

It can be rewritten in the form

$$T_{GM}=\frac{e^\gamma}{\pi}\left(\frac{2}{e}\right)^{7/3}\mu_1 F(\kappa)\exp\left(-\frac{1}{\lambda}\right)$$

$$\equiv 0.277\frac{p_F^2}{2M}F(\kappa)\exp(-\pi\hbar/2p_F|a|),\tag{19}$$

where we expressed the Fermi energy of light atoms μ_2 through the heavy-atom Fermi energy $\mu_1=p_F^2/2M$, and the function $F(\kappa)$ is given by

$$F(\kappa)=\sqrt{\kappa}\exp\left\{-\frac{1+\ln 4}{3}[f(\kappa)+f(\kappa^{-1})-1]\right\}.\tag{20}$$

For equal masses one has $\kappa=M/m=1$, and Eqs. (17) and (20) give $f(1)=1/2$, $F(1)=1$. Then Eq. (18) reproduces the original result of Ref. [30]. The function $F(\kappa)$ is shown in Fig. 2.

For a large mass ratio $\kappa=M/m\gg 1$, this function tends to a constant value, $F(\kappa\to\infty)=2^{4/3}e^{1/6}$. The critical temperature is then given by

$$T_{GM}=\frac{8e^{\gamma-2}}{\pi}2^{2/3}e^{-1/6}\mu_1\exp(-1/\lambda)$$

$$\equiv 0.825\frac{p_F^2}{2M}\exp(-\pi\hbar/2p_F|a|),\quad\kappa\gg 1.\tag{21}$$

Note that this result is quite different from the BCS critical temperature of Eq. (9). Aside from a constant of the order of unity, it contains an extra small factor $1/\sqrt{\kappa}=\sqrt{m/M}\ll 1$.
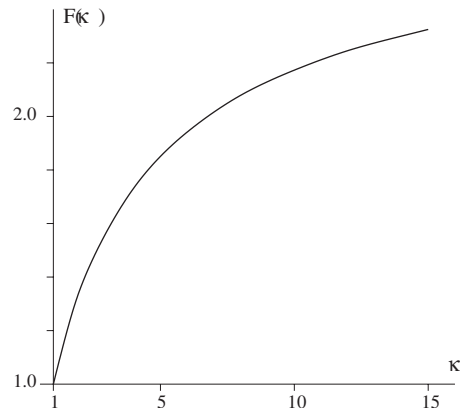

FIG. 2. The function $F(\kappa)$ in the preexponential factor of Eq. (19).

TABLE I. Critical temperatures T_{GM} for mixtures of various atomic species as given by Eq. (19). The scattering length a and Fermi momentum p_F are assumed to be the same for all combinations, and T_{GM} is in units of the critical temperature for a ^{6}Li-^{6}Li mixture.

	^{6}Li	^{40}K	^{87}Sr	^{171}Yb
^{6}Li	1.000	0300	0.161	0.090
^{40}K		0.150	0.097	0.062
^{87}Sr			0.069	0.048
^{171}Yb				0.035

Thus, for a fermionic mixture with a large mass ratio of the components the second-order contribution significantly reduces the critical temperature compared to the prediction of the simple BCS approach.

In Table I we show the critical temperature T_{GM} following from Eq. (19) with $F(\kappa)$ given by Eq. (20), for various mixtures of fermionic atoms. The critical temperature is given in units of T_{GM} for a ^{6}Li-^{6}Li mixture, and it is assumed that the quantity $p_F^2 \exp(-\pi\hbar/2p_F|a|)$ is the same for all mixtures. One clearly sees that replacing one species in a mixture by a lighter one increases the critical temperature, whereas the replacement with a heavier one decreases T_{GM}.

IV. CRITICAL TEMPERATURE. HIGHER ORDER CONTRIBUTIONS

Let us now consider the contribution of higher order $(\sim \lambda^3)$ many-body corrections. As can be seen from Eq. (18), these corrections enter the exponent for the critical temperature being divided by λ^2 and, therefore, the corresponding term is $\lambda Q(M/m)$, where Q is a function of the mass ratio. For moderate values of M/m, the function Q is of the order of unity, and, therefore, the corresponding corrections can be neglected. However, as we will see later, for a large mass ratio, the function Q becomes proportional to M/m, and the related corrections in the exponent are $\sim k_F a M/m$. The applicability of the perturbation theory requires $(k_F a)^2 M/m \ll 1$ [see Eq. (39) below], but the quantity $k_F a M/m$ should not necessarily be small for $M/m \gg 1$. As a result, the third-

order many-body corrections proportional to M/m have to be taken into account.

For calculating the terms of the order of λ^3, we have to modify the gap equation (2) in order to include the difference between particles and quasiparticles, or single-particle excitations [38]. In the considered case of a two-component Fermi gas, the quasiparticles are characterized by the effective masses M^* and m^* and by the Z-factors Z_M and Z_m. The Z factors are related to the amplitude of creating a quasiparticle by adding an extra particle to the system (see [1] for rigorous definitions and details). For a given (effective) interaction V_{eff} between heavy and light fermions, the interaction between the corresponding quasiparticles is simply equal to $Z_M Z_m V_{eff}$. It is important for our approach that the ratios M^*/M, m^*/m, and the constants Z_M, Z_m differ from unity only by a small amount proportional λ.

We can now extend the analysis of the gap equation to higher order terms. As follows from the previous discussions, the qualitative form of the gap equation can be written as

$$1 = \nu_F^* Z_M Z_m V_{eff} \left[\ln \frac{\mu}{T_c} + C \right], \tag{22}$$

where $\nu_F^* = m_r^* k_F / \pi^2 \hbar^2$ and the reduced mass m_r^* is determined by the effective masses M^* and m^*. The effective interaction $V_{eff} = g + \delta V + \delta V^{(3)}$ now includes also the third-order many-body contribution $\delta V^{(3)}$. We are interested in the terms of the order of λ^2 and $\lambda^3 \ln \mu/T_c$ on the right-hand side of Eq. (22), where the $\lambda^3 \ln \mu/T_c$ contributions come from the integration over momenta near the Fermi surface, whereas the momenta far from the Fermi surface result in λ^2 contributions. The term of the order λ^2 in Eq. (22) comes only from the second-order term δV in the effective interaction. For a large mass ratio this term contains only $\ln(M/m)$ for large mass ratio [see Eqs. (15) and (17)] and, therefore, can be neglected. The term $\lambda^3 \ln \mu/T_c$ results from the third-order term $\delta V^{(3)}$ in the effective interaction (with $\nu_F^* Z_M Z_m \rightarrow \nu_F$) and from the difference between particles and quasiparticles, $(\nu_F^* Z_M Z_m - \nu_F) \sim \lambda^2$, multiplied by the first-order term g in the effective interaction. As a result, up to terms of the order of λ^2, we can write the linearized renormalized gap equation as

$$\Delta(\mathbf{p}) = -\frac{2\pi\hbar^2 a}{m_r} \frac{m_r^*}{m_r} Z_M Z_m \int \frac{d\mathbf{p}'}{(2\pi\hbar)^3} \left[\frac{\tanh[\xi_1(\mathbf{p}')/2T_c] + \tanh[\xi_2(\mathbf{p}')/2T_c]}{2[\xi_1(\mathbf{p}') + \xi_2(\mathbf{p}')]} - \frac{2m_r}{p'^2} \right] \Delta(\mathbf{n}' p_F) - \int_{p < \Lambda p_F} \frac{d\mathbf{p}'}{(2\pi\hbar)^3} [\delta V(\mathbf{p}, \mathbf{n}' p_F)$$

$$+ \delta V^{(3)}(\mathbf{p}, \mathbf{n}' p_F)] \frac{\tanh[\xi_1(\mathbf{p}')/2T_c] + \tanh[\xi_2(\mathbf{p}')/2T_c]}{2[\xi_1(\mathbf{p}') + \xi_2(\mathbf{p}')]} \Delta(\mathbf{n}' p_F). \tag{23}$$

As in Eq. (12), we introduce an upper cutoff Λp_F for the purpose of convergence of integrals at large momenta. Equation (23) can be solved in the same way as Eq. (12) and we obtain

$$T_c = \frac{8}{\pi} e^{\gamma-2} \sqrt{\mu_1 \mu_2} \exp\left\{ -\left[\frac{m_r^*}{m_r} Z_M Z_m \lambda - \nu_F (\overline{\delta V} + \overline{\delta V^{(3)}}) \right]^{-1} \right\}$$

$$\approx T_{BCS} \exp\left\{ -\frac{1}{\lambda^2} \left[\left(\frac{m_r^*}{m_r} Z_M Z_m - 1 \right) \lambda + \nu_F (\overline{\delta V} + \overline{\delta V^{(3)}}) \right] \right\}$$

$$= T_{GM} \exp\left\{ -\frac{1}{\lambda^2} \left[\left(\frac{m_r^*}{m_r} Z_M Z_m - 1 \right) \lambda + \nu_F \overline{\delta V^{(3)}} \right] \right\}. \quad (24)$$

Note that only the contributions that are linear in M/m for $M/m \gg 1$ should be kept in m_r^*/m_r, Z_M, Z_m, and $\overline{\delta V^{(3)}}$.

The effective masses M^*, m^* and the constants Z_M, Z_m can be obtained from the derivatives of the corresponding self-energies $\Sigma_M(\omega, p)$ and $\Sigma_m(\omega, p)$ with respect to the frequency ω and momentum p, evaluated at $\omega=0$ and $p=p_F$,

$$Z_{M(m)} = \left(1 - \left. \frac{\partial \Sigma_{M(m)}(\omega, p)}{\partial \omega} \right|_{\omega=0, p=p_F} \right)^{-1}, \quad (25)$$

$$M^*/M = Z_M^{-1} \left(1 + \frac{M}{p_F} \left. \frac{\partial \Sigma_M(\omega, p)}{\partial p} \right|_{\omega=0, p=p_F} \right)^{-1}, \quad (26)$$

$$m^*/m = Z_m^{-1} \left(1 + \frac{m}{p_F} \left. \frac{\partial \Sigma_m(\omega, p)}{\partial p} \right|_{\omega=0, p=p_F} \right)^{-1}. \quad (27)$$

The diagrams for the self-energies Σ_M and Σ_m up to the second order in g are shown in Fig. 3.

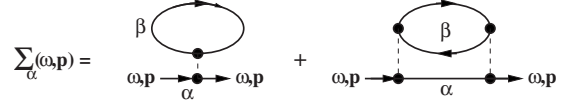The corresponding analytical expressions read

FIG. 3. The first- and second-order diagrams to the self-energy $\Sigma_\alpha(\omega, p)$. Note that we present here only irreducible diagrams, and, therefore, the diagram of the second order containing the first-order self-energy insertion to the Green's function of the β fermion in the first diagram is omitted.

$$\Sigma_\alpha(\omega, p) = \frac{2\pi\hbar^2 a}{m_r} n_\beta + \left(\frac{2\pi\hbar^2 a}{m_r} \right)^2 \int \frac{d\omega_2}{2\pi} \frac{d\mathbf{p}_2}{(2\pi\hbar)^3} G_\beta(\omega_2, \mathbf{p}_2)$$

$$\times \int \frac{d\omega_1}{2\pi} \frac{d\mathbf{p}_1}{(2\pi\hbar)^3} [G_\beta(\omega_1 + \omega_2, \mathbf{p}_2 + \mathbf{p}_1) G_\alpha(-\omega_1$$

$$+ \omega, -\mathbf{p}_1 + \mathbf{p}) - G_\beta^{(0)}(\omega_1, \mathbf{p}_1) G_\alpha^{(0)}(-\omega_1, -\mathbf{p}_1)], \quad (28)$$

where $\alpha=M, \beta=m$ or $\alpha=m, \beta=M$, and the Green's functions are given by

$$G_{\alpha(\beta)}(\omega, \mathbf{p}) = \frac{1}{\omega - \xi_{\alpha(\beta)}(\mathbf{p}) + i\delta \, \text{sgn}[\xi_{\alpha(\beta)}(\mathbf{p})]},$$

$$G_{\alpha(\beta)}^{(0)}(\omega, \mathbf{p}) = \frac{1}{\omega - p^2/2m_{\alpha(\beta)} + i\delta}$$

with $\delta = +0$. The divergent integral in the second-order contribution is renormalized in a standard way by replacing the coupling constant g with the scattering amplitude a and subtracting the product $G_\beta^{(0)}(\omega_1, \mathbf{p}_1) G_\alpha^{(0)}(-\omega_1, -\mathbf{p}_1)$ of the two Green's functions in vacuum ($\mu_{1,2}=0$), which corresponds to the second-order Born contribution to the scattering amplitude, from the integrand. After integrating over the frequencies ω_1 and ω_2 in Eq. (28), we obtain

$$\Sigma_\alpha(\omega, p) = \frac{2\pi\hbar^2 a}{m_r} n + \left(\frac{2\pi\hbar^2 a}{m_r} \right)^2 \int \frac{d\mathbf{p}_2}{(2\pi\hbar)^3} \frac{d\mathbf{p}_1}{(2\pi\hbar)^3} \left\{ \frac{(1 - f[\xi_\beta(\mathbf{p}_2)]) f[\xi_\alpha(-\mathbf{p}_1 + \mathbf{p})] f[\xi_\beta(\mathbf{p}_2 + \mathbf{p}_1)]}{\omega - \xi_\alpha(-\mathbf{p}_1 + \mathbf{p}) - \xi_\beta(\mathbf{p}_2 + \mathbf{p}_1) + \xi_\beta(\mathbf{p}_2) - i\delta} \right.$$

$$\left. + \frac{f[\xi_\beta(\mathbf{p}_2)](1 - f[\xi_\alpha(-\mathbf{p}_1 + \mathbf{p})])(1 - f[\xi_\beta(\mathbf{p}_2 + \mathbf{p}_1)])}{\omega - \xi_\alpha(-\mathbf{p}_1 + \mathbf{p}) - \xi_\beta(\mathbf{p}_2 + \mathbf{p}_1) + \xi_\beta(\mathbf{p}_2) + i\delta} + \frac{f[\xi_\beta(\mathbf{p}_2)]}{p_1^2/2m_r - i\delta} \right\}. \quad (29)$$

As we discussed above, for a large M/m only the leading contributions that are linear in M/m should be kept, and lengthy calculations with the use of Eqs. (25)–(29) give

$$Z_M = 1, \quad (30)$$

$$\frac{M^*}{M} = 1 + \frac{1}{5}(2\ln 2 - 1)\left(\frac{a p_F}{\pi\hbar} \right)^2 \frac{M}{m}, \quad (31)$$

$$Z_m = 1 - \frac{1}{3}(1 + 2\ln 2)\left(\frac{a p_F}{\pi\hbar} \right)^2 \frac{M}{m}, \quad (32)$$

$$\frac{m^*}{m} = 1 + \frac{1}{3}(1 + 2\ln 2)\left(\frac{a p_F}{\pi\hbar} \right)^2 \frac{M}{m}. \quad (33)$$

The diagrams for third-order contributions to the effective interaction $V_{\text{eff}}(\mathbf{p}, \mathbf{p}')$ are shown in Fig. 4, where we omit the diagrams that can be obtained by inserting the first-order self-energy blocks (the first diagram in Fig. 3) into the internal lines of the second-order diagram from Fig. 1. These self-energy contributions [the first term in Eq. (28)] simply shift the chemical potentials. It turns out that only diagrams 4(a)–4(c) could contain terms linear in M/m, whereas the
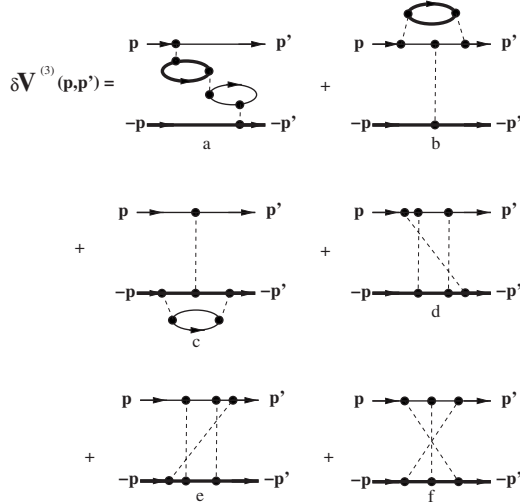
FIG. 4. The third-order contributions to the effective interaction between heavy (thick line) and light (thin line) fermions. The dashed line corresponds to the coupling constant g.

rest of the diagrams are proportional to $\ln(M/m)$. The divergencies at large momenta in diagrams 4(d) and 4(f) can be removed by renormalizing the coupling constant g in the

second-order diagram in Fig. 1. Analytical expressions for the diagrams 4(a)–4(c) are the following:

$$\delta V_a^{(3)}(\mathbf{p}, \mathbf{p}') = g^3 \int \frac{d\omega_1 d\mathbf{p}_1}{(2\pi)^4 \hbar^3} G_M(\omega_1, \mathbf{q} + \mathbf{p}_1)$$

$$\times G_M(\omega_1, \mathbf{p}_1) \int \frac{d\omega_2 d\mathbf{p}_2}{(2\pi)^4 \hbar^3}$$

$$\times G_m(\omega_2, \mathbf{q} + \mathbf{p}_2) G_m(\omega_2, \mathbf{p}_2),$$

$$\delta V_b^{(3)}(\mathbf{p}, \mathbf{p}') = g^3 \int \frac{d\omega_1 d\mathbf{p}_1}{(2\pi)^4 \hbar^3} G_M(\omega_1, \mathbf{p} + \mathbf{p}_1) G_M(\omega_1, \mathbf{p}' + \mathbf{p}_1)$$

$$\times \int \frac{d\omega_2 d\mathbf{p}_2}{(2\pi)^4 \hbar^3} G_m(\omega_2, \mathbf{p}_2) G_m(\omega_1 + \omega_2, \mathbf{p}_1 + \mathbf{p}_2),$$

$$\delta V_c^{(3)}(\mathbf{p}, \mathbf{p}') = g^3 \int \frac{d\omega_1 d\mathbf{p}_1}{(2\pi)^4 \hbar^3} G_m(\omega_1, \mathbf{p} + \mathbf{p}_1) G_m(\omega_1, \mathbf{p}' + \mathbf{p}_1)$$

$$\times \int \frac{d\omega_2 d\mathbf{p}_2}{(2\pi)^4 \hbar^3} G_M(\omega_2, \mathbf{p}_2) G_M(\omega_1 + \omega_2, \mathbf{p}_1 + \mathbf{p}_2),$$

where $\mathbf{q} = \mathbf{p} - \mathbf{p}'$. The integration over frequencies ω_1 and ω_2 in the above expressions is straightforward and gives

$$\delta V_a^{(3)}(\mathbf{p}, \mathbf{p}') = g^3 \nu_M \nu_m \int \frac{d\mathbf{p}_1}{(2\pi\hbar)^3} \frac{f[\xi_1(\mathbf{p}_1 + \mathbf{q})] - f[\xi_1(\mathbf{p}_1)]}{\xi_1(\mathbf{p}_1 + \mathbf{q}) - \xi_1(\mathbf{p}_1)} \int \frac{d\mathbf{p}_2}{(2\pi\hbar)^3} \frac{f[\xi_2(\mathbf{p}_2 + \mathbf{q})] - f[\xi_2(\mathbf{p}_2)]}{\xi_2(\mathbf{p}_2 + \mathbf{q}) - \xi_2(\mathbf{p}_2)}$$

$$= g^3 \nu_M \nu_m \frac{1}{4} \left[1 + \frac{p_F}{q} \left(1 - \frac{q^2}{4p_F^2} \right) \ln \frac{2p_F + q}{|2p_F - q|} \right]^2,$$

where $\nu_M = M p_F / 2\pi^2 \hbar^3$ and $\nu_m = m p_F / 2\pi^2 \hbar^3$ are the densities of states at the Fermi level for heavy and light fermions, respectively. For the other two contributions we obtain

$$\delta V_b^{(3)}(\mathbf{p}, \mathbf{p}') = g^3 \int \frac{d\mathbf{p}_1}{(2\pi\hbar)^3} \int_0^\infty ds A_M(s, \mathbf{p}_1) \left[\frac{f[\xi_2(\mathbf{p}_1 + \mathbf{p})]}{\xi_2(\mathbf{p}_1 + \mathbf{p}) - \xi_2(\mathbf{p}_1 + \mathbf{p}') + i\delta} \frac{1}{\xi_2(\mathbf{p}_1 + \mathbf{p}) - s} \right.$$

$$\left. + \frac{1 - f[\xi_2(\mathbf{p}_1 + \mathbf{p})]}{\xi_2(\mathbf{p}_1 + \mathbf{p}) - \xi_2(\mathbf{p}_1 + \mathbf{p}') - i\delta} \frac{1}{\xi_2(\mathbf{p}_1 + \mathbf{p}) + s} + (\mathbf{p} \leftrightarrow \mathbf{p}') \right]$$

with

$$A_M(s, \mathbf{p}) = \frac{M p_F^2}{8\pi^2 \hbar^3 p} \begin{cases} 1 - \left(\dfrac{Ms}{pp_F} - \dfrac{p}{2p_F} \right)^2, & \dfrac{p}{2M} |p - 2p_F| \leq s \leq \dfrac{p}{2M} (p + 2p_F) \\ \dfrac{2Ms}{p_F^2}, & 0 \leq s \leq \dfrac{p}{2M} (2p_F - p); \quad p \leq 2p_F, \end{cases} \tag{34}$$

and zero otherwise. A similar expression is obtained for $\delta V_c^{(3)}(\mathbf{p}, \mathbf{p}')$, with the replacements $\xi_2 \to \xi_1$ and $A_M(s, \mathbf{p}) \to A_m(s, \mathbf{p})$.

The corresponding contributions to the s-wave scattering channel can be obtained by averaging over the directions of the momenta $\mathbf{p}$ and $\mathbf{p}'$,

$$\overline{\delta V_j^{(3)}} = \int \frac{d\hat{\mathbf{p}}}{4\pi} \int \frac{d\hat{\mathbf{p}}'}{4\pi} \delta V_j^{(3)}(\mathbf{p}, \mathbf{p}'), \quad j = a, b, c.$$

In the limit of $M/m \gg 1$, the leading terms in these contributions are

$$\overline{\delta V_a^{(3)}} = g \frac{2 + 7\zeta(3)}{16} \left(\frac{a p_F}{\pi \hbar} \right)^2 \frac{M}{m}, \tag{35}$$

$$\overline{\delta V_b^{(3)}} = 0, \tag{36}$$

$$\overline{\delta V_c^{(3)}} = -g \frac{1 + 4(2 + 3\ln 2)\ln 2}{18} \left(\frac{a p_F}{\pi \hbar} \right)^2 \frac{M}{m}, \tag{37}$$

where $\zeta(x)$ is the Riemann zeta functions $[\zeta(3) = 1.202]$. As a result, the quantity $\overline{\delta V^{(3)}}$ in Eq. (24) is

$$\overline{\delta V^{(3)}} = \overline{\delta V_a^{(3)}} + \overline{\delta V_b^{(3)}} + \overline{\delta V_c^{(3)}}. \tag{38}$$

Note that the validity of the perturbation theory requires the quantity $\overline{\delta V^{(3)}}$ be smaller than the coupling constant g. This leads to the condition

$$(a p_F / \hbar)^2 M/m \ll 1. \tag{39}$$

Thus, the actual small parameter of the theory in the limit of a large mass ratio M/m is $(p_F |a|/\hbar)\sqrt{M/m}$.

After substituting Eqs. (30)–(33) and (38) into Eq. (24) we find the critical temperature in the limit of $M/m \gg 1$,

$$T_c = \frac{8 e^{\gamma-2}}{\pi} 2^{2/3} e^{-1/6} \frac{p_F^2}{2M} \exp\left(-\frac{\pi \hbar}{2|a|p_F} - 0.034 \frac{|a|p_F}{\pi \hbar} \frac{M}{m} \right)$$

$$= T_{GM} \exp\{-0.011(p_F |a|/\hbar)M/m\}. \tag{40}$$

Compared to the transition temperature in the GM approach, Eq. (40) contains an extra exponential factor which, in principle, can be large. However, this requires a very high mass ratio M/m. The extra term in the exponent of Eq. (40) can be written as $0.01\sqrt{M/m} \times (p_F |a|/\hbar)\sqrt{M/m}$ and, since the second multiple in this expression is small, one should have the mass ratio at least of the order of thousands in order to obtain a noticeable change of T_c compared to the GM result. In this case Eq. (39) shows that the parameter $\lambda = 2 p_F |a|/\pi \hbar$ should be very small and, hence, the transition temperature itself is vanishingly low.

We thus see that for reasonable values of $p_F |a|/\hbar$ satisfying Eq. (39), let us say $p_F |a|/\hbar \sim 0.1$ and $M/m < 100$, the higher-order contributions do not really change the GM result for the transition temperature.

At the same time, our analysis shows that for $M/m \gg 1$ the conventional weakly interacting regime requires much lower values of $\lambda = 2 p_F |a|/\pi \hbar$ than in the case of equal masses and the small parameter of the perturbation theory is given by Eq. (39) In the next section we discuss the physical origin of this parameter.

V. SMALL PARAMETER OF THE THEORY

There are several conditions that allow one to develop a perturbation theory for a many-body fermionic system on the basis of Hamiltonian (1). First of all, this is the condition of the weakly interacting regime, which assumes that the amplitude a of the interspecies interaction is much smaller than the mean separation between particles. The latter is of the order of $\hbar/p_F$, and we immediately have the inequality

$$p_F |a|/\hbar \ll 1. \tag{41}$$

At the same time, inequality (41) allows one to use the binary approach for the interparticle interaction. Then, assuming a short-range character of the interatomic potential, one can consider the interaction between particles as contact and write the interaction part of the Hamiltonian as $g \int d\mathbf{r} \hat{\psi}_1^+(\mathbf{r}) \hat{\psi}_1(\mathbf{r}) \hat{\psi}_2^+(\mathbf{r}) \hat{\psi}_2(\mathbf{r})$.

In the weakly interacting regime only fermions near the Fermi surface participate in the response of the system to external perturbations. Therefore, there is another condition that is needed for constructing the perturbation theory. Namely, we have to assume that for both light and heavy fermions the density of states near the Fermi surface is not strongly distorted by the interactions. This is certainly the case if both Fermi energies, $p_F^2/2m$ and $p_F^2/2M$, greatly exceed the mean-field interaction ng. For $M \sim m$ this condition is equivalent to inequality (41). In contrast, for $M \gg m$ Eq. (41) only guarantees that the Fermi energy of light fermions is $p_F^2/2m \gg ng$, whereas the condition $p_F^2/2M \gg ng$ leads to the inequality $(p_F |a|/\hbar)M/m \ll 1$. This mean-field condition, however, is far too strong because at the mean-field level, the interaction shifts uniformly all energy states and, hence, results only in the change of the chemical potential. Actually, the interaction-induced modification of the density of states is determined by the momentum and frequency dependence of the fermionic self-energy [see Eqs. (25)–(27)] and appears in the second order of the perturbative expansion in g (the second diagram in Fig. 3). The corresponding contribution describes the process in which a heavy fermion pushes a light one out of the Fermi sphere and then, interacting once more with this light fermion, puts it back to the initial state. Due to the Pauli principle, the momenta of both light and heavy fermions in the intermediate state should be larger than the Fermi momentum. As a result, for the initial heavy-fermion state close to the Fermi surface, the most important intermediate states will be those with momenta close to the Fermi momentum. Therefore, the resulting contribution should be proportional to the product of the densities of
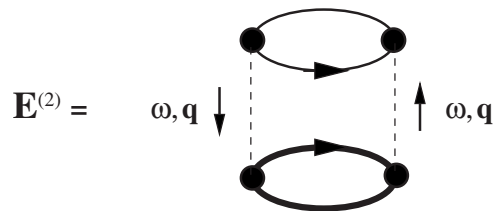

FIG. 5. The second-order contribution to the energy of the system.

states of heavy and light fermions at the Fermi surface, and the relative change of both densities of states is controlled by the parameter $g^2 \nu_M \nu_m \sim (p_F a/\hbar)^2 M/m$. Thus, this parameter should be small, i.e., we arrive at Eq. (39),

$$(p_F a/\hbar)^2 M/m \ll 1.$$

A complementary physical argument on support of this small parameter comes from the consideration of the effective interaction between a light and a heavy fermion in the medium. For example, the process described by the diagram in Fig. 4(a) can be viewed in the following way. Incoming heavy and light fermions interact with fermions inside the filled Fermi spheres and transfer them to the states above the Fermi surfaces. Then the transferred heavy and light fermions interact with each other and return to their initial states. The important point is that the intermediate state of this process contains excitations (particle-hole pairs) near the Fermi surface of the filled Fermi sphere of heavy fermions. Therefore, the corresponding contribution to the effective interaction is $g_{eff} \sim g^3 \nu_m \nu_M$, where the densities of states of heavy and light fermions near the Fermi surface are $\nu_M = M p_F/2\pi^2 \hbar^3$ and $\nu_m = m p_F/2\pi^2 \hbar^3$, respectively. This leads to $g_{eff} \sim g(p_F a/\hbar)^2 M/m$. Comparing it with the direct interaction g and requiring the inequality $|g_{eff}| \ll |g|$ which allows one to use a perturbation theory, we again obtain a small parameter of the theory $(p_F a/\hbar)^2 M/m \ll 1$.

Let us now understand in which physical quantities the parameter (39) enters directly. In the limit of $M/m \gg 1$ heavy fermions occupy the energy interval $p_F^2/2M$ which is much narrower than the energy interval $p_F^2/2m$ occupied by light fermions. However, the heavy-fermion density of states is much larger: $\nu_M = M p_F/(2\pi^2\hbar^3) \gg \nu_m = m p_F/(2\pi^2\hbar^3)$. The high density of states of heavy fermions manifests itself in any quantity characterized by processes where, for the heavy fermions, only the states near the Fermi surface are important. This is the case for the effective masses of atoms, critical temperature, and (see the next section) for the zero-temperature order parameter Δ_0. If, however, all states of the heavy fermions are important, then the peak at $E \sim p_F^2/2M$ in the energy distribution of heavy fermions is integrated out, and the result does not contain the parameter (39). This is exactly what is happening in the calculation of the second-order correction to the energy of the system, which involves the sum over all energy states.

The second-order contribution to the energy is shown diagrammatically in Fig. 5, and the corresponding analytical expression reads

$$E^{(2)} = g^2 \int \frac{d\omega}{2\pi} \frac{d\mathbf{q}}{(2\pi\hbar)^3} \Pi_m(\omega, q) \Pi_M(-\omega, q), \quad (42)$$

where

$$\Pi_\alpha(\omega, q) = \int \frac{d\omega_1}{2\pi} \frac{d\mathbf{p}_1}{(2\pi\hbar)^3} G_\alpha(\omega + \omega_1, \mathbf{q} + \mathbf{p}_1) G_\alpha(\omega_1, \mathbf{p}_1) = -\int \frac{d\mathbf{p}_1}{(2\pi\hbar)^3}$$

$$\times \frac{f[\xi_\alpha(\mathbf{p}_1 + \mathbf{q})] - f[\xi_\alpha(\mathbf{p}_1)]}{\omega - [\xi_\alpha(\mathbf{p}_1 + \mathbf{q}) - \xi_\alpha(\mathbf{p}_1)] + i\delta\{\mathrm{sgn}[\xi_\alpha(\mathbf{p}_1 + \mathbf{q})] - \mathrm{sgn}[\xi_\alpha(\mathbf{p}_1)]\}}$$

is the polarization operator (the bubble in the diagrammatic language) for α fermions ($\alpha = M$ or m). As it can be seen, the integration over ω in Eq. (42) results in an integral that diverges at large q. This divergence, however, can be eliminated by subtracting the second-order Born contribution to the interparticle scattering amplitude multiplied by the densities of fermions. This corresponds to the renormalization of the coupling constant g in the first-order (mean-field) contribution to the energy $E^{(1)} = gn^2$ (see [1] for more details). The resulting expression then coincides with Eq. (6.12) in [1].

After using the spectral representation for the polarization operator $\Pi_\alpha(\omega, p)$,

$$\Pi_\alpha(\omega, q) = \int_0^\infty ds A_\alpha(s, q) \left[\frac{1}{\omega - s + i\delta} - \frac{1}{\omega + s - i\delta} \right],$$

with the function $A_\alpha(s, p)$ from Eq. (34), Eq. (42) can be rewritten in the form

$$E^{(2)} = \frac{1}{2} g^2 \int \frac{d\mathbf{q}}{(2\pi\hbar)^3} \left[\int_0^\infty ds_1 \int_0^\infty ds_2 \frac{A_M(s_1, q) A_m(s_2, q)}{s_1 + s_2} - \frac{2m_r}{q^2} n^2 \right], \quad (43)$$

where the second term in the brackets corresponds to the renormalization. In the limit of $M/m \gg 1$, as follows from Eq. (34), typical values of s_1 are much smaller than typical values of s_2. We therefore can neglect s_1 in the denominator of the first term in Eq. (43) and replace m_r by m. This gives

$$E^{(2)} = \frac{1}{2}g^2 \int \frac{d\mathbf{q}}{(2\pi\hbar)^3}\left[\int_0^\infty ds_1 A_M(s_1,q)\int_0^\infty ds_2 \frac{A_m(s_2,q)}{s_2} - \frac{2m}{q^2}n^2\right] = \frac{1}{2}g^2 \int \frac{d\mathbf{q}}{(2\pi\hbar)^3}\left[\int_0^\infty ds_1 A_M(s_1,q)\left(-\frac{1}{2}\right)\Pi_m(0,q) - \frac{2m}{q^2}n^2\right]$$

$$= \frac{1}{2}g^2 \int \frac{d\mathbf{q}}{(2\pi\hbar)^3}\left\{n\left[\theta(q-2p_F) + \frac{3}{4}\frac{q}{p_F}\left(1 - \frac{q^2}{12p_F^2}\right)\theta(2p_F - q)\right]\left(-\frac{1}{2}\right)\Pi_m(0,q) - \frac{2m}{q^2}n^2\right\}$$

$$= \frac{1}{2}g^2 \int \frac{d\mathbf{q}}{(2\pi\hbar)^3}\left\{n\left[\theta(q-2p_F) + \frac{3}{4}\frac{q}{p_F}\left(1 - \frac{q^2}{12p_F^2}\right)\theta(2p_F - q)\right]\frac{\nu_m}{4}\left[1 + \frac{p_F}{q}\left(1 - \frac{q^2}{4p_F^2}\right)\ln\frac{2p_F+q}{|2p_F-q|}\right] - \frac{2m}{q^2}n^2\right\}$$

$$= gn^2 \frac{9(8\ln 2 - 9)}{140}\frac{ap_F}{\pi\hbar}.$$

As we see, the final result does not depend on M, as it was anticipated above.

VI. ORDER PARAMETER AND SINGLE-PARTICLE EXCITATIONS

We now calculate the order parameter and its temperature dependence. At zero temperature no quasiparticles are present since both quasiparticle energies are positive ($E_\pm > 0$). Then, confining ourselves to second-order terms in g, the renormalized gap equation reads

$$\Delta_0(\mathbf{p}) = -\frac{2\pi\hbar^2 a}{m_r}\int\frac{d\mathbf{p}'}{(2\pi\hbar)^3}$$
$$\times\left[\frac{1}{E_+(\mathbf{p}') + E_-(\mathbf{p}')} - \frac{2m_r}{p'^2}\right]\Delta_0(\mathbf{p}')$$
$$-\int\frac{d\mathbf{p}'}{(2\pi\hbar)^3}\delta V(\mathbf{p},\mathbf{p}')\frac{1}{E_+(\mathbf{p}') + E_-(\mathbf{p}')}\Delta_0(\mathbf{p}').$$
$$(44)$$

Strictly speaking, the many-body contribution to the interparticle interaction δV is affected by the superfluid pairing and, therefore, does not coincide with that of Eq. (11). However, at zero temperature the difference is proportional to Δ_0/μ_i and, hence, is exponentially small. Therefore, we can use Eq. (11) for δV in Eq. (44).

The arguments used above for obtaining Eq. (19) from Eq. (5), can also be applied here, but the large logarithm $\ln(\mu/T_c)$ should be replaced by $\ln[\mu/\Delta_0(p_F)]$.

For the value of the order parameter at the Fermi surface, $\Delta_0(p_F)$, we obtain

$$\Delta_0(p_F) = \left(\frac{2}{e}\right)^{7/3}\exp\left\{-\frac{1+\ln 4}{3}[f(\kappa) + f(\kappa^{-1})\right.$$
$$\left. - 1]\right\}\frac{p_F^2}{4m_r}\exp\left(-\frac{1}{\lambda}\right).$$
$$(45)$$

Comparing Eq. (18) with Eq. (45) we obtain a relation between $\Delta_0(p_F)$ and T_c,

$$T_c = \frac{e^\gamma}{\pi}\frac{2}{\kappa^{1/2} + \kappa^{-1/2}}\Delta_0(p_F). \quad (46)$$

We should emphasize that relation (45) between the order parameter and the critical temperature remains valid after taking into account higher order terms in the gap equation, which is necessary for a large mass ratio (see the previous section for the discussion of the critical temperature). The generalization of Eq. (44) in order to include the higher order terms repeats the derivation of Eq. (23) and the resulting equation reads

$$\Delta(\mathbf{p}) = -\frac{2\pi\hbar^2 a}{m_r}\frac{m_r^*}{m_r}Z_M Z_m \int \frac{d\mathbf{p}'}{(2\pi\hbar)^3}\left[\frac{1}{E_+(\mathbf{p}') + E_-(\mathbf{p}')}\right.$$
$$- \frac{2m_r}{p'^2}\right]\Delta(\mathbf{n}'p_F) - \int_{p<\Lambda p_F}\frac{d\mathbf{p}'}{(2\pi\hbar)^3}[\delta V(\mathbf{p},\mathbf{n}'p_F) + \delta V^{(3)}$$
$$\times(\mathbf{p},\mathbf{n}'p_F)]\frac{1}{E_+(\mathbf{p}') + E_-(\mathbf{p}')}\Delta(\mathbf{n}'p_F). \quad (47)$$

This equation can be solved in a way similar to that of solving Eq. (23), and the solution is

$$\Delta_0(p_F) = \pi e^{-\gamma}\frac{p_F^2}{4m_r}\exp\left\{-\left[\frac{m_r^*}{m_r}Z_M Z_m\lambda - \nu_F(\overline{\delta V}\right.\right.$$
$$\left.\left. + \overline{\delta V^{(3)}})\right]^{-1}\right\}. \quad (48)$$

Comparing Eq. (24) with Eq. (48), we immediately obtain Eq. (46).

We now analyze the order parameter in the two limiting cases: $M = m$ and $M \gg m$. In the case of equal masses we have $m_r = m/2$ and recover the usual expression [30] for the order parameter from Eq. (45),

$$\Delta_0(p_F) = \left(\frac{2}{e}\right)^{7/3}\frac{p_F^2}{2M}\exp\left(-\frac{\pi\hbar}{2p_F|a|}\right) = 1.764 T_c; \quad M = m.$$

In the case of $M \gg m$, i.e., $\kappa \gg 1$, we have $m_r \approx m$ and, including higher order contributions, from Eqs. (46) and (21) we obtain
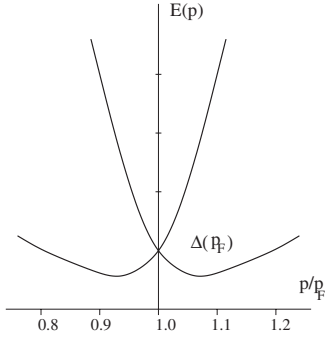
FIG. 6. Two branches $E_\pm(p)$ (in arbitrary units) of single-particle excitations for $M/m=10$.

$$\Delta_0(p_F) = 2^{8/3} e^{-13/6} \frac{p_F^2}{2\sqrt{Mm}} \exp\left(-\frac{\pi\hbar}{2p_F|a|} - 0.011\frac{|a|p_F}{\hbar}\frac{M}{m}\right)$$

$$= 0.882\sqrt{\frac{M}{m}} T_c \gg T_c; \quad M \gg m. \quad (49)$$

Note that in this limit the order parameter at the Fermi surface is much larger than the critical temperature.

In order to analyze the behavior of the order parameter Δ for temperatures close to the critical temperature, $(T_c - T) \ll T_c$, we have to expand the gap equation (4) in powers of $\Delta/T_c \ll 1$ and keep the cubic term. The result can be written as

$$\Delta(p_F)\left[\ln\frac{T_c}{T} - \frac{4\kappa}{(1+\kappa)^2}\frac{7\zeta(3)}{8\pi^2}\left(\frac{\Delta(p_F)}{T_c}\right)^2\right] = 0.$$

From this equation we obtain

$$\Delta(p_F) = \sqrt{\frac{8\pi^2}{7\zeta(3)}}\frac{\kappa^{1/2}+\kappa^{-1/2}}{2} T_c \sqrt{1-\frac{T}{T_c}}. \quad (50)$$

In the limit of equal masses, $\kappa=1$, we reproduce the well-known result for the temperature dependence of the order parameter. In the opposite limit of a large mass ratio, $\kappa \gg 1$, we find

$$\Delta(p_F) = \sqrt{\frac{M}{m}}\sqrt{\frac{8\pi^2}{7\zeta(3)}} T_c \sqrt{1-\frac{T}{T_c}}. \quad (51)$$

Note that as well as Eq. (49), the obtained equation (50) contains a large factor $\sqrt{M/m}$.

At any temperature the energies of single-particle excitations are given by [see Eq. (3) and Fig. 6]

$$E_\pm(p) = \pm\left(\frac{p^2-p_F^2}{4m_-}\right) + \sqrt{\left(\frac{p^2-p_F^2}{4m_r}\right)^2 + \Delta^2(p_F)}, \quad (52)$$

where $m_- = Mm/(M-m)$.

Equation (52) reveals a peculiar feature of fermionic mixtures with unequal masses of components. For equal masses, the minimum of $E_\pm(\mathbf{p})$ (i.e., the gap) is at the Fermi surface and equals $\Delta(p_F)$. The situation for unequal masses is different: the single-particle excitation energies $E_\pm(p)$ reach their minimum values $E_{\pm\min}=\Delta(p_F)\sqrt{1-s^2}=2\Delta(p_F)/(\kappa^{1/2}+\kappa^{-1/2})$ at momenta $p_\pm^{*2}=p_F^2 \mp 2m\kappa^{1/2}(\kappa-1)\Delta(p_F)/(\kappa+1)$. For M/m
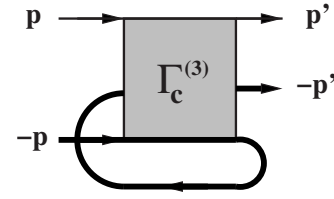

FIG. 7. The contribution of three-body processes between one light and two heavy fermions described by the connected three-body vertex $\Gamma_c^{(3)}$, to the Gorkov-Melikh-Barkhudarov corrections.

$\gg 1$ the corresponding gap is much smaller than $\Delta(p_F)$,

$$E_{\pm\min}(M \gg m) \approx 2\Delta(p_F)\kappa^{-1/2} = 2\sqrt{\frac{m}{M}}\Delta(p_F) \ll \Delta(p_F). \quad (53)$$

It is interesting to note that the presence of a small factor $\sqrt{m/M}$ in Eq. (53) restores the intuitive picture that the gap in the single-particle spectrum and the critical temperature are of the same order of magnitude even in the limit of a large mass ratio. We point out, however, that in this limit the order parameter on the Fermi surface, being much larger than the critical temperature, is not equal to the gap in the single-particle spectrum. This gap is of the order of the critical temperature, and the low-energy single-particle excitations correspond to momenta different from the Fermi momentum p_F. Owing to the former circumstance, one does not expect any dramatic changes in thermodynamic properties of the system with increasing the mass ratio M/m to a large value.

VII. THREE-BODY RESONANCES

Let us now discuss the influence of the three-body physics on the results of the previous sections. The diagram for the Gorkov-Melikh-Barkhudarov corrections in Fig. 1 corresponds to collisions between three particles: two from a Cooper pair and one from the filled Fermi sea. Thus, it is a three-body process. During this process, however, the three particles undergo two successive two-body collisions and never appear simultaneously within the range of the interatomic interaction. The corresponding three-body wave function vanishes when the hyperspherical radius [see the definition before Eq. (54)] is tending to zero.

In a dilute two-component Fermi gas, real three-body collisions during which the three colliding particles simultaneously approach each other, are rare. An additional smallness compared to a Bose gas is provided by the Pauli principle. Two of the three colliding particles are identical fermions and, therefore, the wave function of their relative motion should strongly decrease at small separations. As a result, the contribution of such collisions to the effective pairing interaction is small and can be neglected. However, for the case of a large mass ratio M/m the situation is more subtle. If $M/m > 13.6$, two heavy and one light fermions can form three-body bound states [24,33,40]. The most interesting case corresponds to the presence of a weakly bound trimer state because this results in a resonance three-body scattering at low energy. It is not clear that these resonances
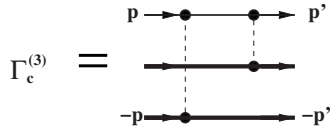
FIG. 8. The lowest order contribution to the connected three-body vertex.

should be taken into account when calculating the pairing energy since there may be nontrivial issues of wave function statistics involved; nevertheless we shall estimate their possible contribution and leave such issues for further study [39].

To analyze the effect of three-body bound states we note that the contribution to the Gorkov-Melikh-Barkhudarov corrections in Fig. 1 is part of a more general contribution involving the connected three-body vertex function $\Gamma_c^{(3)}$ (see Fig. 7).

This is a consequence of a general relation between two- and three-particle Green's functions. The quantity $\Gamma_c^{(3)}(\{p_i\}_{\mathrm{in}}, \{p_i'\}_{\mathrm{out}})$ with $p_i = (\omega_i, \mathbf{p}_i)$, $i = 1, 2, 3$, describes the scattering of two heavy and one light particle from the initial state with incoming energy-momenta p_i into the final state with outgoing energy-momenta p_i'. For $\omega_i = p_i^2/2m$ (the mass-shell condition) the vertex function coincides with the T matrix. By definition, the connected vertex function $\Gamma_c^{(3)}$ does not include three-body processes in which only two out of the three particles collide (in our case, a light fermion collides with only one heavy fermion) and, therefore, $\Gamma_c^{(3)}$ is represented only by connected diagrams. The general three-body vertex function $\Gamma^{(3)}$ [see Eq. (56)] contains all diagrams including disconnected ones. Those describe processes in which only two out of the three particles interact with each other. Fig. 8 shows the simplest contribution to $\Gamma_c^{(3)}$ that is second order and results in the Gorkov-Melikh-Barkhudarov corrections shown in Fig. 1.

We consider the case where the size of a three-body bound state is much larger than $|a|$, but much smaller than the average distance between particles in the gas. Accordingly, the binding energy is much larger than typical kinetic energies of particles, the Fermi energies μ_i. In this case, the influence of other particles of the gas can be neglected and the properties of the bound state can be found by solving the three-body Schrödinger equation. Introducing the hyperspherical radius $\rho = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$ in the six-dimensional space $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}\sqrt{(2M+m)/4m} = \mathbf{r}_1 - (\mathbf{r}_2 + \mathbf{r}_3)/2$ is the distance between the light fermion and the center of mass of two heavy ones separated from each other by a distance $\mathbf{y} = \mathbf{r}_2 - \mathbf{r}_3$, the normalized wave function of a shallow bound state with the binding energy $E_b = -\hbar^2/2Mb^2$ and the size $b \gg |a|$, has the form [41]

$$\varphi(\rho) \sim \left(\frac{m}{M}\right)^{3/4} \frac{1}{|a|} \times \begin{cases} \dfrac{\Phi_1(\Omega)}{\rho^2}, & \rho \ll |a|, \\[2mm] \dfrac{|a|^3 \Phi_2(\Omega)}{\rho^5}, & a \ll \rho \ll b. \end{cases} \tag{54}$$

Here Φ_1 and Φ_2 are the functions of hyperangles Ω, and we do not give explicit expressions for these functions because

of their complexity. For $\rho > b$, the wave function decays exponentially. Note that the normalization of the wave function (54) is determined by distances $\rho \sim |a|$.

Most conveniently the contribution of the bound state to the vertex function can be found using the three-body Green's function $G(\{\mathbf{p}_i\}, \{\mathbf{p}_i'\}, \omega)$,

$$G^{(3)}(\{\mathbf{p}_i\}, \{\mathbf{p}_i'\}, \omega) = \left\langle \{\mathbf{p}_i'\} \left| \frac{1}{\omega - H + i0} \right| \{\mathbf{p}_i\} \right\rangle,$$

where the Hamiltonian H has the form $H = H_0 + \hat{V}$ with

$$H_0 = \frac{p_1^2}{2m} + \frac{1}{2M}(p_2^2 + p_3^2)$$

and

$$\hat{V} = g\delta(\mathbf{r}_1 - \mathbf{r}_2) + g\delta(\mathbf{r}_1 - \mathbf{r}_3)$$

in the coordinate representation (index 1 corresponds to the light fermion and indices 2 and 3 to the heavy ones). The Green's function satisfies the equation

$$HG^{(3)}(\{\mathbf{p}_i\}, \{\mathbf{p}_i'\}, \omega) = \prod_{i=1,2,3} \delta(\mathbf{p}_i - \mathbf{p}_i'),$$

which is equivalent to the integral equation

$$G^{(3)} = G_0^{(3)} + G_0^{(3)} \hat{V} G^{(3)}, \tag{55}$$

with $G_0^{(3)} = [\omega - p_1^2/2m - (p_2^2 + p_3^2)/2M + i0]^{-1} \Pi_i \delta(\mathbf{p}_i - \mathbf{p}_i')$ being the Green's function for free particles. This equation can be rewritten in the form

$$G^{(3)} = G_0^{(3)} + G_0^{(3)} \Gamma^{(3)} G_0^{(3)}, \tag{56}$$

where we introduce the vertex function $\Gamma^{(3)}$. This function describes all scattering processes involving three particles, both connected (described by $\Gamma_c^{(3)}$) and disconnected ones (not included in $\Gamma_c^{(3)}$). The vertex function $\Gamma^{(3)}$ obeys the Lipmann-Schwinger equation

$$\Gamma^{(3)} = \hat{V} + \hat{V} G_0^{(3)} \Gamma^{(3)} \tag{57}$$

and, as it can be seen from Eqs. (55)–(57), is related to the Green's function $G^{(3)}$ as

$$\Gamma^{(3)} = \hat{V} + \hat{V} G^{(3)} \hat{V}. \tag{58}$$

It is convenient to use the spectral decomposition of the Green's function. In the center-of-mass reference frame, where $\Sigma_i \mathbf{p}_i = \Sigma_i \mathbf{p}_i' = 0$, this decomposition reads

$$G^{(3)}(\{\mathbf{p}_i\}, \{\mathbf{p}_i'\}, \omega) = \sum_n \Psi_n^*(\{\mathbf{p}_i\}) \frac{1}{\omega - E_n + i0} \Psi_n(\{\mathbf{p}_i'\})$$

$$+ \int d\lambda \Psi_\lambda^{(+)*}(\{\mathbf{p}_i\}) \frac{1}{\omega - E_\lambda + i0} \Psi_\lambda^{(+)}(\{\mathbf{p}_i'\}), \tag{59}$$

where the summation is performed over a complete set $\{\Psi_n, \Psi_\lambda^{(+)}\}$ of eigenfunctions of the three-body Hamiltonian H with eigenenergies E_n, E_λ, respectively. The eigenfunctions Ψ_n correspond to bound states with energies $E_n < 0$,

and the eigenfunctions $\Psi_\lambda^{(+)}$ to scattering states of three particles with energies $E_\lambda > 0$. Their asymptotic behavior at large interparticle distances contains incoming plane waves with momenta specified by the index λ, and outgoing (therefore, index $+$) scattered waves. These eigenfunctions vanish for small hyperspherical radius ρ and, in particular, they describe the Gorkov-Melik-Barkhudarov corrections discussed above. On the contrary, the bound state eigenfunctions are nonzero for small ρ and decay exponentially for $\rho \to \infty$.

Equations (58) and (59) together give the decomposition of the vertex function $\Gamma^{(3)}$ in terms of the solutions of the three-body Schrödinger equation. Obviously, the bound states contribute only to the connected vertex function $\Gamma_c^{(3)}$. In particular, the contribution of the bound state Ψ_n is

$$\delta_n\Gamma^{(3)}(\{\mathbf{p}_i\},\{\mathbf{p}_i'\},\omega) = \delta_n\Gamma_c^{(3)}(\{\mathbf{p}_i\},\{\mathbf{p}_i'\},\omega)$$
$$= [\hat{V}\Psi_n(\{\mathbf{p}_i\})]^* \frac{1}{\omega - E_n + i0}\hat{V}\Psi_n(\{\mathbf{p}_i'\}). \tag{60}$$

By using the Schrödinger equation

$$(H_0 + \hat{V})\Psi_n = E_n\Psi_n,$$

Eq. (60) can be rewritten in the form

$$\delta_n\Gamma_c^{(3)}(\{\mathbf{p}_i\},\{\mathbf{p}_i'\},\omega)$$
$$= \Psi_n^*(\{\mathbf{p}_i\})\frac{[E_n - E_0(\{\mathbf{p}_i\})][E_n - E_0(\{\mathbf{p}_i'\})]}{\omega - E_n + i0}\Psi_n(\{\mathbf{p}_i'\}), \tag{61}$$

where $E_0(\{\mathbf{p}_i\}) = p_1^2/2m + (p_2^2 + p_3^2)/2M$.

Now we can estimate the contribution of the weakly bound state of one light and two heavy fermions to the effective interparticle interaction V_{eff} between light and heavy fermions with opposite momenta on the Fermi surface. The analytical expression corresponding to the diagram in Fig. 7 is

$$\delta V_{\text{eff}} = \int \frac{d\omega}{2\pi}\int\frac{d\mathbf{q}}{(2\pi\hbar)^3}\Gamma_c^{(3)}(\mathbf{p},\mathbf{q},-\mathbf{p};\mathbf{p}',-\mathbf{p}',\mathbf{q};\omega)$$
$$\times \frac{1}{\omega - (q^2 - p_F^2)/2M + i0\,\text{sgn}(q - p_F)}. \tag{62}$$

The contribution of the bound state is then obtained by substituting Eq. (61) into Eq. (62) and integrating out the frequency ω,

$$\delta_n V_{\text{eff}} = \int\frac{d\mathbf{q}}{(2\pi\hbar)^3}\theta(p_F - q)\Psi_n^*(\mathbf{p},\mathbf{q},-\mathbf{p})$$
$$\times\frac{[E_n - \mu_1 - \mu_2 - q^2/2M]^2}{(q^2 - p_F^2)/2M + \mu_1 + 2\mu_2 - E_n - q^2/[2(2M + m)]}$$
$$\times\Psi_n(\mathbf{p}',-\mathbf{p}',\mathbf{q}), \tag{63}$$

where the last term in the denominator corresponds to the motion of the center of mass. In the considered case, the binding energy E_n is the largest energy scale ($E_n \gg \mu_1, \mu_2$), and $\delta_n V_{\text{eff}}$ in Eq. (63) can be estimated as

$$\delta_n V_{\text{eff}} \sim (p_F/\hbar)^3 E_b|\Psi_n(0,0,0)|^2, \tag{64}$$

where the factor $(p_F/\hbar)^3$ results from the integration over $d\mathbf{q}$ and we used the condition $|a|p_F/\hbar \ll 1$ to set all momenta in the wave function of the bound state to zero. After using the wave function from Eq. (54), we obtain

$$\Psi_n(0,0,0) \sim (p_F/\hbar)\left(\frac{M}{m}\right)^{3/4}(p_F/\hbar)a^2b^2,$$

and therefore

$$\nu_F\delta_n V_{\text{eff}} \sim \nu_F(p_F/\hbar)^5 E_b a^4 b^4\left(\frac{M}{m}\right)^{3/2}$$
$$\sim (p_F|a|/\hbar)^4(p_F b/\hbar)^2\sqrt{\frac{M}{m}}.$$

This result has to be compared with the GM contribution $\nu_F\overline{\delta V} \sim (p_F|a|/\hbar)^2\ln(M/m)$, and with the contribution of third-order terms $\nu_F\delta V^{(3)} \sim (p_F|a|/\hbar)^3 M/m$. Under the condition $(p_F b/\hbar) < 1$ corresponding to a not too shallow bound state, we find that the contribution of three-body resonances is small compared to both of them,

$$\frac{\nu_F\delta_n V_{\text{eff}}}{\nu_F\overline{\delta V}} \sim (p_F|a|/\hbar)^2(p_F b/\hbar)^2\frac{\sqrt{M/m}}{\ln(M/m)} \ll 1,$$

and

$$\frac{\nu_F\delta_n V_{\text{eff}}}{\nu_F\delta V^{(3)}} \sim (p_F|a|/\hbar)(p_F b/\hbar)^2\sqrt{m/M} \ll 1.$$

We thus see that three-body resonances are rather narrow, and their contribution to the effective interaction can be omitted. So, the results obtained in the previous sections for the critical temperature, effective masses, order parameter, and elementary excitations remain unchanged.

VIII. CONCLUDING REMARKS

We now give an outlook on the physics of attractively interacting mixtures of heavy and light fermionic atoms in view of the results obtained in this paper. We have developed a perturbation theory in the BCS limit for the heavy-light superfluid pairing along the lines proposed by Gorkov and Melik-Barkhudarov [30] and found that for $M/m \gg 1$ one has to take into account both the second-order and third-order contributions. The result for the critical temperature and order parameter is then quite different from the outcome of the simple BCS approach. Moreover, the small parameter of the theory is given by Eq. (39) and reads $(p_F|a|/\hbar)\sqrt{M/m} \ll 1$. As we explained in Sec. V, this can be seen from the second-order correction to the fermionic self-energy, which is controlled by the parameter $g^2\nu_M\nu_m \sim (p_F a/\hbar)^2 M/m$. Therefore, in a mixture of heavy and light fermions the conventional perturbation theory for the weakly interacting regime requires much smaller p_F (densities) and/or $|a|$ than in the case of $M \sim m$, where the small parameter is $(p_F|a|/\hbar) \ll 1$.

Let us now discuss the cases of $M = m$ and $M \gg m$ regarding the regimes of superfluid pairing. For $M = m$ we have the
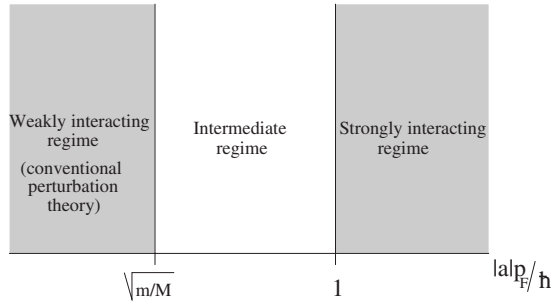
FIG. 9. Regimes of superfluid pairing for $M/m \gg 1$. In the intermediate regime the conventional perturbation theory is not applicable (see text).

strongly interacting regime for $(p_F|a|/\hbar) \gtrsim 1$, and the BCS limit for $(p_F|a|/\hbar) \ll 1$. In the former case the perturbation theory is not applicable and the results are obtained either by Monte Carlo methods or by adjusting the mean-field theory to this regime (see [5] for review). For $M \gg m$ the situation is different. As we found, the conventional perturbation theory works well under the condition $(p_F|a|/\hbar) \ll \sqrt{m/M}$ (see Fig. 9). For $(p_F|a|/\hbar) \gtrsim 1$ we have the strongly interacting regime where the perturbation theory does not work at all. However, we now have a range of densities and scattering lengths, where $\sqrt{m/M} \ll (p_F|a|/\hbar) \ll 1$. In this intermediate regime one can still use Hamiltonian (1) and try to develop a perturbative approach, since the scattering amplitude is much smaller than the mean interparticle separation. On the other hand, the conventional perturbation theory does not work for the reasons explained in Sec. V. In order to construct a reliable theory one should at least renormalize the interaction between heavy and light fermions by making an exact resummation of diagrams containing loops of heavy and light fermions. We then expect a substantial renormalization of the properties of the superfluid phase.

We thus see that our findings pave a way to revealing types of superfluid pairing in mixtures of attractively interacting ultracold fermionic atoms with very different masses. An appropriate candidate is a gaseous mixture of ^{171}Yb with ^{6}Li, and one should work out possibilities for tuning the Li-Yb interaction in this system. Another candidate is a two-species system of fermionic atoms in an optical lattice with a small filling factor. The difference in the hopping amplitudes of the species can be made rather large, which corresponds to a large ratio of the "heavy" to "light" effective mass. For example, in the case of ^{6}Li-^{40}K mixture one can increase the mass ratio by a factor of 20 in a lattice with period of 250 nm and the tunneling rates $\sim 10^3$ s^{-1} and $\sim 10^5$ s^{-1} for K and Li, respectively.

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