

Optimal purification of thermal graph states

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Abstract. In this paper, a purification protocol is presented and its performance is proven to be optimal when applied to a particular subset of graph states that are subject to local Z -noise. Such mixed states can be produced by bringing a system into thermal equilibrium, when it is described by a Hamiltonian which has a particular graph state as its unique ground state. From this protocol, we derive the exact value of the critical temperature T_{crit} above which purification is impossible, as well as the related optimal purification rates. A possible simulation of graph Hamiltonians is proposed, which requires only bipartite interactions and local magnetic fields, enabling the tuning of the system temperature.

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1. Introduction

Any quantum technological implementation is plagued by environmental noise. The possibility to purify quantum states, or to use error correcting algorithms to stabilize quantum operations, is therefore a necessary step towards reaping the benefits of quantum technologies. Much attention has been focused lately on the purification [1]–[7] of a large class of multipartite states called graph states [8]. After the initial restriction to two colourable graph states [4, 5], the ideas have been extended to all graph states [7], and generalized to other stabilizer states [9]. The variety of different protocols trade off between a large tolerance to noise [4, 5] and the rate of purification [6]. Graph states have proven important for realizing a variety of quantum information tasks such as performing quantum computation [10], quantum communication [11] and as a means for efficiently approximating other quantum states, such as the ground states of strongly correlated systems [12]. While some bounds on the ability to purify multipartite states have previously been proven, optimal results only exist for two-qubit states [15, 16].

Here, we concentrate on the purification of graph states that are subject to the physically motivated independent Z -noise. From the technological perspective, large-scale quantum computation is still too difficult to implement. However, graph states can be made and manipulated in the laboratory, e.g. by controlled collisions of alkali atoms trapped in optical lattices [17, 18]. To enable the controlled collisions, it is necessary to employ two magnetically sensitive hyperfine levels of the atoms to form a qubit. This sensitivity means that the states are subject to decoherence from stray magnetic fields, in addition to any uncontrolled collisions that may occur. These errors are described by Z -errors, and as experiments improve, one can expect them to become more localized. An alternative approach to preparing a graph state involves implementing a Hamiltonian, known as a graph Hamiltonian, which has the desired state as its ground state. As it is impossible to cool the system to absolute zero, the resulting equilibrium state will always be a thermal state. For graph Hamiltonians, the thermal noise corresponds to local Z -noise on each qubit.

In this paper, we consider a certain purification protocol applied to arbitrary graph states in the presence of independent Z -errors. While this multipartite protocol is not novel or sophisticated [1, 13, 19], it has the advantage of being analytically tractable. Most importantly, we prove optimality of this protocol for these types of error, both in the sense of the level of noise that can be tolerated as well as the scaling of the purification rate, for a specific subset of states that includes the cluster and Greenberger–Horne–Zeilinger (GHZ) states. For up to seven qubits, this

subset can be shown to be isomorphic to arbitrary graphs under local operations. We propose a method for simulating the graph Hamiltonians that is comprised of only two-body collisions and local magnetic fields, even though the resulting Hamiltonian has at least three-qubit interactions [21, 20]. The critical purification temperature related to these models can easily be made to lie above the typical temperatures given, e.g. from optical lattice realizations of such Hamiltonians.

2. Graph Hamiltonians and graph states

Let us introduce a graph G , which is defined by a set of vertices V_G , and a set of edges E_G , describing the connections between the vertices. To each vertex of this graph, we attach a spin- $\frac{1}{2}$ particle (qubit), and define a graph state to be the ground state of the following Hamiltonian,

$$H = -\frac{1}{2} \sum_{i \in V_G} B_i K_i, \quad K_i = X_i \prod_{\{i,j\} \in E_G} Z_j. \quad (1)$$

The B_i 's are the coupling strengths, which we henceforth take to have equal magnitude, $B_i = B$, and we assume $B > 0$. The interaction terms K_i commute with each other, $(K_i, K_j) = 0$, and hence each term individually stabilizes the eigenstates of H . A local Pauli Z -rotation on qubit i , Z_i , commutes with all the K_j where $j \neq i$, and anticommutes with K_i . Hence, the excitations of the Hamiltonian are given by local Z -rotations applied to the ground state.

A constructive way to produce graph states is found by close analogy with the cluster states [8, 21]. We can produce a general graph state by creating the $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ state on each vertex of G , and applying controlled phase gates along the edges of the graph. The action of measurements on the cluster state also applies to the graph states, such that when a Z -measurement is performed, all nearest-neighbour bonds are severed. This means that graph states can be cut into sections using Z -measurements, and can be combined together using controlled phase gates. The key to what follows is the realization that Z -errors commute with both of these operations, and consequently remain as Z -errors.

Examples of graph states that are of particular interest in quantum computation and communication are the cluster [22] and GHZ [23] states. The first state corresponds to a graph that is given by a d -dimensional cube, while the latter state corresponds to a graph that is locally equivalent to a single qubit connected with all other qubits.

3. Purification of thermal graph states

The thermal state of the Hamiltonian in equation (1) of $|V_G| = N$ qubits at temperature $T = 1/(k_B\beta)$ is given by

$$\rho = \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})}, \quad (2)$$

where we set the Boltzmann constant k_B equal to unity. This density matrix can be written in terms of local Z -errors as $\rho(p) = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_N |\psi\rangle \langle \psi|$, where

$$\mathcal{E}_i \rho = [(1 - p)\rho + pZ_i \rho Z_i]$$

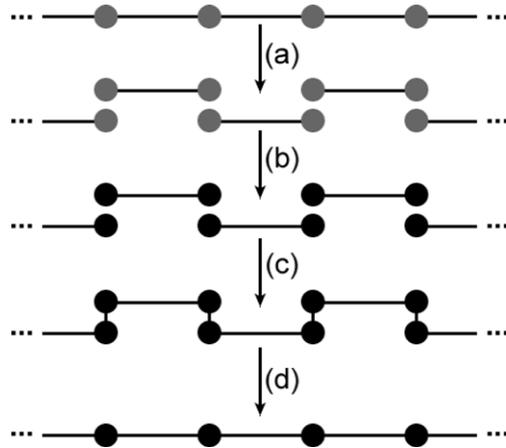


Figure 1. Purification protocol for independent noise. First, we take many copies of the noisy graph state. (a) We construct from these two qubit nearest-neighbour states (noisy). (b) Two-qubit states are purified (if possible). (c) Controlled phase gates are applied between local qubits. (d) All qubits except one from each party are measured in the x -basis, leaving the remaining qubits in the purified state.

and

$$p = \frac{1}{1 + e^{\beta B}} \quad (3)$$

is the probability of a Z -error occurring at a certain site due to the nonzero temperature T . The graph state $|\psi\rangle$ is the unique ground state of Hamiltonian (1). Our aim is to purify towards the state $|\psi\rangle$ using many copies of $\rho(p)$. We consider that each vertex of the graph is controlled by a different party and that operations such as measurements and controlled phase gates are only allowed locally, but involving the many copies. This restriction corresponds to the scenario of quantum repeaters [24], where the different parties are physically separated, and also serves to illustrate the entanglement properties of the system. We do not envisage implementation of the protocol in other scenarios, since it is generally cheaper in terms of resources to create the state directly—our aim is to prove optimality, providing a benchmark for all other protocols.

The purification procedure consists of breaking down the graph states into smaller blocks, purifying them, and then recombining them. We refer to it as the Divide and Rebuild Purification Protocol (DRPP). The smaller blocks that we choose to use are two-qubit states, for which there are analytic purification results, returning maximally entangled states. As already specified, the splitting of the graph state is readily achieved with Z -measurements. This leaves us with a two-qubit mixed state $\rho_2(p)$, which can be purified to a maximally entangled state $|\psi_2\rangle$ provided

$$\langle \psi_2 | \rho_2(p) | \psi_2 \rangle > \frac{1}{2}.$$

Once we have generated a maximally entangled state for each of the edges of the graph, each local party (i) holds a number of qubits equal to the number of nearest neighbours $|E_G^i|$. These can be reduced to a single qubit that contains all these links by applying controlled phase gates and performing x -measurements, as described in figure 1.

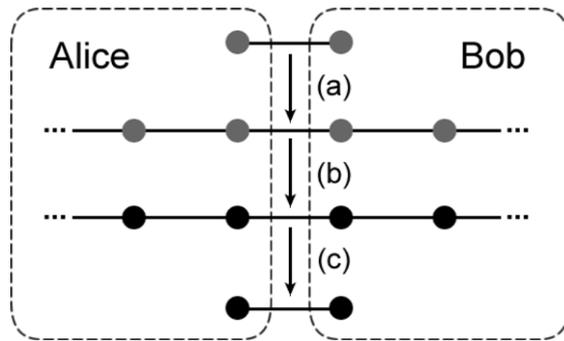


Figure 2. If we assume the multipartite state can be purified, then this implies that we can purify the two-qubit state. Conversely, if the two-qubit state cannot be purified, the assumption must be broken. (a) Alice and Bob take the two-qubit state and reconstruct the noisy graph state. (b) This state is purified by assumption. (c) All extra qubits are measured out to return the original pair, now pure.

The condition for purification of the two-qubit state between neighbouring sites is readily found to be

$$(1 - p)^2 > \frac{1}{2}. \quad (4)$$

This is known to be necessary and sufficient for purifying states that are Bell diagonal, like the states considered here [15, 16, 24]. We can hence calculate that the maximum temperature at which the DRPP can purify graph states is given by

$$T_{\text{crit}} = \frac{-B}{\ln(\sqrt{2} - 1)}. \quad (5)$$

4. Optimality and rates

While the DRPP can be applied to any graph state (with minor modifications), we can prove its optimality for a specific subset of graphs. For clarity, we shall restrict to only cluster states (of arbitrary dimension) and GHZ states, all of which fall into this classification. The optimality that we prove is with respect to both the level of noise that can be tolerated such that purification is still possible, and with respect to the number of copies of the initial, noisy, state required to form a single pure copy. In particular, one can prove that states with higher levels of noise than the one dictated by equation (4) can never be purified with *any* protocol. We do this by considering a purification protocol for a two-qubit state. The two parties sharing this state are allowed to introduce extra qubits and the local operations that they apply subsume multipartite considerations for the additional qubits.

If two parties, Alice and Bob, hold several copies of the noisy two-qubit state $\rho_2(p)$,⁴ they can locally recreate the initial thermal state, $\rho(p)$. For linear graphs (figure 2), this simply

⁴ Note that all the noisy two-qubit states are identical.

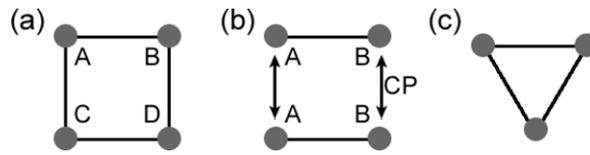


Figure 3. (a) Simplest example of a $2d$ -cluster state, shared between four parties. (b) Alice and Bob can locally reconstruct the square graph using two copies of ρ_2 , and applying controlled phase gates between them. (c) For the triangular configuration, the optimality proof fails.

corresponds to Alice and Bob locally creating their own thermal cluster states, and connecting them to $\rho_2(p)$ with controlled phase gates. In the case of a more general graph, we require some additional connections between Alice and Bob. These are achieved by using multiple copies of $\rho_2(p)$ (figure 3(b)). At this stage, we assume that purification of $\rho(p)$ is possible, yielding $|\psi\rangle$. From there, Alice and Bob can measure out all the qubits that they added, leaving a pure two-qubit state. Hence, if $\rho(p)$ can be purified, $\rho_2(p)$ can always be purified. However, we know that $\rho_2(p)$ cannot be purified if $(1 - p)^2 \leq \frac{1}{2}$. Hence, under this condition, our assumption must be false i.e. the multipartite state $\rho(p)$ cannot be purified, whatever the protocol. The DRPP saturates this bound and hence is optimal for independent Z -noise. While analysis of the protocol of [4, 5] is a difficult problem, numerical results indicate that it also saturates the bound, and hence is optimal for this type of noise.

The rate of purification, R_ψ , of the graph state $|\psi\rangle$ can be calculated in terms of the yield of a Bell state, R_2 . We take the standard definition of rate,

$$R_\psi = \frac{\text{Copies of } |\psi\rangle \text{ produced}}{\text{Copies of } \rho \text{ consumed}}.$$

If we can purify ρ into $|\psi\rangle\langle\psi|$ at a rate R_ψ (i.e. we require $1/R_\psi$ copies of ρ to create $|\psi\rangle$), then we can create a Bell state between any linked pair just by performing Z -measurements on excess qubits. There could be a more efficient way to generate this Bell pair, requiring fewer copies, so $R_\psi \leq R_2$.

Similarly, if we can purify Bell pairs, then we can generate $|\psi\rangle$. For that we take a Bell pair between each nearest neighbour and the local parties perform the reconstruction as specified by the DRPP (e.g. controlled phase between local states and x -measurements). Most of the Bell pairs can be purified in parallel—we only require N_{geo} copies of ρ to purify enough copies, where the geometric factor N_{geo} depends only on the local degree of the graph, $|E_G^i|$, and is otherwise independent of the number of qubits in the graph. This is because the Z -measurements commute with the errors, and divide the state ρ into separate blocks. As a result, $R_\psi \geq (R_2/N_{\text{geo}})$. Combining the two results,

$$R_2 \geq R_\psi \geq \frac{R_2}{N_{\text{geo}}}.$$

For d -dimensional cluster states, we can readily evaluate N_{geo} , since we must measure out all qubits connected directly to the pair that we are interested in isolating. Starting from the edge of the lattice, this uses up $3d - 2$ qubits (we include in this number some single qubits that, while we might not intend to measure them, become isolated), plus the two for the state ρ_2 . If our lattice extends to N qubits in each direction, then we can generate on average $(N - 1)N^{d-1}/(3d)$

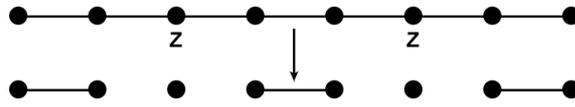


Figure 4. By measuring in the Z basis on every third qubit, we can create two-qubit states between all nearest neighbours with only three copies of the original linear cluster state. Hence, we say $N_{\text{geo}} = 3$.

copies of ρ_2 from a single copy of ρ . We need $d(N-1)N^{d-1}$ different copies of ρ_2 , and hence $N_{\text{geo}} = 3d^2$. As demonstrated in figure 4, this corresponds to $N_{\text{geo}} = 3$ for $d = 1$. The resulting rate is independent of the number of qubits in the system, only depending on R_2 , which is optimal up to a small numerical factor. For N -qubit GHZ states, $N_{\text{geo}} = N - 1$.

5. Physical implementation

A significant achievement in recent quantum engineering experiments is the construction of cluster states with optical lattices [17]. They are produced with a single operational step independent of the size of the system. It is natural to consider this setup for studying the purification properties of thermal cluster states. This requires the implementation of both the cluster Hamiltonian and a purification protocol in a physical setup. In the following, we will present a simple way of simulating the Hamiltonian (1) for cluster states, using proven experimental techniques in optical lattices.

When an entangled state decoheres, there is a characteristic lifetime that determines when the state becomes separable. On the other hand, if it is possible to obtain an interaction described by the Hamiltonian that has this entangled state as a ground state, then provided the energy gap is large enough in comparison to the decoherence rate, entanglement can survive indefinitely in the system, e.g. in the form of purifiable mixed states, as we have already seen.

Simulating Hamiltonian (1) for a general graph is a relatively straightforward task. The method we adopt here consists of a unitary operation, $U_G^\dagger [U_G]$, applied before (after) the evolution with respect to a local Hamiltonian. This evolution is generated by applying a uniform magnetic field in the x -direction. When U_G corresponds to controlled phase gates between all pairs of qubits connected in the graph, then it is easy to show that the resulting effective Hamiltonian

$$H = U_G \left(B \sum_i X_i \right) U_G^\dagger \quad (6)$$

is of the form (1) [25]. The spectrum of the Hamiltonian corresponding to the magnetic field $B \sum_i X_i$ is the same as that of the Hamiltonian H , as they are related by an isospectral transformation. Hence, the generated thermal state directly corresponds to the one of H .

In optical lattices, the unitary U_G can be realized in cubic lattices by controlled collisions between nearest neighbours. This is precisely the operation which is experimentally employed in [17] for the generation of a cluster state. Meanwhile, the local magnetic fields are implemented by globally applied Raman transitions between the hyperfine states that encodes a qubit. Thus, the realization of the graph Hamiltonian is readily achieved. The stationary state of this

system is the thermal state of equation (2). Further, this system gives us the ability to vary the temperature. In previous experiments [26], the recorded temperature after performing optical cooling was given by the relation $T \approx 0.1U_0$, where U_0 is light shift potential created by the optical lattice. From the value of the critical temperature given in (5), one deduces that by employing moderate local magnetic fields with amplitude $B \gtrsim 0.1U_0$, one can bring the system into the purifiable regime.

Once we are able to implement a purification protocol in optical lattices, whether it be the DRPP or any other, this provides us with the perfect test-bed to probe the maximum temperature that still allows purification, and verify the critical temperature given in equation (5). The potential implementation of such schemes has been described extensively in, for example, [5]. The main drawback is the requirement of local addressability, which can be circumvented with the help of superlattices [18, 27, 28] by breaking of the translational invariance of the lattice.

6. Conclusions

Here, a purification protocol has been proven to be optimal when applied to a subclass of graph states, including the cluster and GHZ states, subject to Z -errors. Although we have restricted ourselves to this particular form of noise in this paper, we emphasize that thermal states arising from graph Hamiltonians are precisely of this form, rendering the considered types of states interesting also from an experimental and practical point of view.

While the optimality proof cannot be applied to certain types of graphs (e.g. the icosahedron), and we are aware of examples of noise (e.g. local or global white noise) where the proposed protocol is not optimal, several extensions of our results are possible. Two of the authors have shown [29] that the optimality proof can be extended to a wider class of states and to different forms of noise. For instance, it can be shown that all graph states of up to seven qubits can be brought by local unitary operations to a form where the optimality proof can be applied [8]. Thus, all these graphs have a critical temperature given by equation (5). In addition, it is shown in [29] how the optimality proof presented in this paper can be extended to other types of noise, providing an upper-bound to the error probability that can be purified. Further extensions taking into account non-graph states and noisy local operations during the purification protocol will also be examined.

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