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## Quantum-limited velocity readout and quantum feedback cooling of a trapped ion via electromagnetically induced transparency

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We discuss continuous observation of the momentum of a single atom by employing the high velocity sensitivity of the index of refraction in a driven  $\Lambda$ -system based on electromagnetically induced transparency. In the ideal limit of unit collection efficiency this provides a quantum-limited measurement with minimal backaction on the atomic motion. A feedback loop, which drives the atom with a force proportional to measured signal, provides a cooling mechanism for the atomic motion. We derive the master equation which describes the feedback cooling and show that in the Lamb-Dicke limit the steady state energies are close to the ground state, limited only by the photon collection efficiency. Outside of the Lamb-Dicke regime the predicted temperatures are well below the Doppler limit.

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### I. INTRODUCTION

Quantum feedback control employs the strategy of acting on a system based on measurement data obtained by continuous observation of the quantum system of interest, thus achieving control of quantum dynamics and preparation of particular quantum states [1–4]. A prerequisite of developing quantum feedback control is the realization of quantum-limited measurements. Quantum optical systems and, more recently, mesoscopic systems have taken a leading role in achieving these requirements towards demonstration of feedback control in the laboratory [5–13].

Motivated by the remarkable experimental progress with trapped ions, we will develop in the present paper a theory of quantum feedback cooling of a single ion, based on a continuous readout of the atomic velocity via *dispersive* interactions with laser light. The idea is to devise a (in principle) quantum-limited measurement of the velocity by employing the strong detuning (and thus velocity) dependence of the index of refraction of a driven  $\Lambda$  system near the atomic dark state. These dark states are coherent superpositions of two atomic ground states which do not couple to the excited atomic state, which leads to strong suppression of dissipative light scattering. This is the same feature which underlies recent studies of electromagnetically induced transparency, slow light in atomic gases, and quantum memory of light in atomic ensembles. The present setup of dispersive readout of the atomic velocity complements, and is in contrast to, ongoing experiments of quantum feedback cooling of a single two-level ion in front of a mirror [14,15], where the position of the ion is continuously monitored by emission of light into the mirror mode, as analyzed theoretically in our recent publication [16].

Our discussion of quantum feedback cooling of a single trapped ion in a strongly driven atomic  $\Lambda$ -system builds on, and connects various well-developed topics in atomic physics

and quantum optics, as well as continuous measurement and quantum feedback theory. Thus we find it worthwhile to present both a brief review of the background material and physical key ideas underlying the present work, as well as the main results of the paper in Sec. II. In Sec. III we give the technical details of our model and derive the equations for the measured signal and the conditioned evolution of the atomic motion for a weakly excited atom. A Wiseman-Milburn-type [1] master equation for feedback cooling and the resulting temperatures will be discussed in Sec. IV. Finally, in Sec. V we make a connection to the EIT-laser cooling and describe the combination of feedback and laser cooling. The details of the adiabatic elimination of the internal atomic states are given in Appendices A and B.

### II. OVERVIEW AND SUMMARY

In this section we present an overview of the concepts and the main results of this paper. Our emphasis will be on explaining the basic physics and strategy behind our quantum feedback scheme, and providing references to later sections where the mathematically inclined reader can find the details of the derivations. In Sec. II A we will briefly review EIT and discuss the dependence of the index of refraction on the ion momentum near the dark state resonance. Continuous read out of the ion momentum using homodyne detection will be formulated in Sec. II B. The main results of the present paper are the equations for feedback cooling in Sec. II C, and the predictions for the final temperatures and cooling rates in II D, in particular, also in connection to the EIT laser cooling of ions [17–19].

#### A. Electromagnetically induced transparency

The phenomenon of EIT is related to a quantum interference effect which in its simplest form can be observed in a

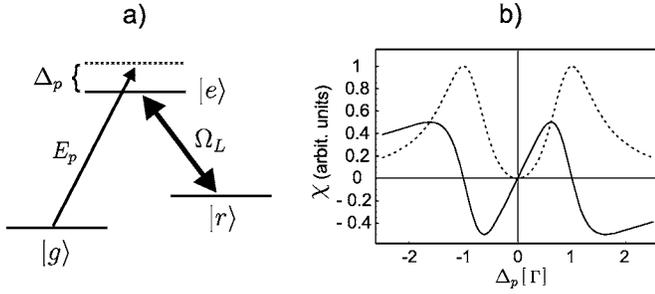


FIG. 1. (a) Atomic  $\Lambda$  system with two ground states  $|g\rangle$  and  $|r\rangle$ . The transition  $|r\rangle \rightarrow |e\rangle$  is driven by a strong resonant light field with Rabi frequency  $\Omega_L$ , while the probe field  $E_p$  couples  $|g\rangle \rightarrow |e\rangle$  with detuning  $\Delta_p$ . (b) Susceptibility  $\chi(\Delta_p)$  of a  $\Lambda$  system as a function of the detuning of the probe field,  $E_p$ . At the center of the transparency window the real part of the susceptibility (solid line) representing the index of refraction of the medium exhibits a steep slope, while the imaginary part (dotted line) vanishes.

three-level atom (for a review see [20,21], and references therein). To discuss this effect we consider an atom with the internal states  $|g\rangle$ ,  $|e\rangle$ , and  $|r\rangle$  in a  $\Lambda$  configuration as shown in Fig. 1. The transition between the states  $|r\rangle$  and  $|e\rangle$  is driven by a strong, resonant laser, while a second light field, the probe field, couples the ground state to the excited state. We denote by  $\Delta_p = \omega_p - \omega_{eg}$  the detuning of the probe field from the atomic resonance. The atomic Hamiltonian is then given by

$$H_\Lambda = \hbar\Delta_p|g\rangle\langle g| + \frac{\hbar}{2}(\Omega_L|e\rangle\langle r| + g|e\rangle\langle g| + \text{H.c.}), \quad (1)$$

where  $\Omega_L$  and  $g$  are the Rabi frequencies of the laser and the probe field, respectively. At the two photon resonance,  $\Delta_p=0$ , the Hamiltonian  $H_\Lambda$  has an adiabatic eigenstate with zero energy, a so-called “dark state,”

$$|D\rangle \sim \Omega_L|g\rangle - g|r\rangle. \quad (2)$$

For an atom in the state  $|D\rangle$ , the excitation from the states  $|g\rangle$  and  $|r\rangle$  destructively interfere and the atom decouples from the light. We note that an atom in a dark state involves no excited state population, and is thus immune to decay from the excited state.

The existence of such a dark state leads to remarkable properties of the index of refraction. For weak probe fields, the propagation can be discussed in terms of the linear susceptibility,  $\chi(\omega_p)$ . The real part of  $\chi(\omega_p)$  is related to the refractive index by  $n = 1 + \text{Re}(\chi(\omega_p))/2$ , while the imaginary part,  $\text{Im}(\chi(\omega_p))$ , is proportional to the absorption coefficient. For an ensemble of three-level atoms where both,  $|g\rangle$  and  $|r\rangle$  are long-lived states, the susceptibility has the characteristic behavior [21],

$$\chi(\Delta_p) \sim \frac{\Delta_p}{|\Omega_L|^2/4 + i\Delta_p(\Gamma + i\Delta_p)}, \quad (3)$$

where  $\Gamma$  is the decay rate of the excited state. Figure 1(b) shows the dependence of  $\chi$  on the detuning,  $\Delta_p$ . Around the dark resonance  $\Delta_p=0$  we have a steep slope of the refractive index (solid line) which leads to a slow group velocity of the

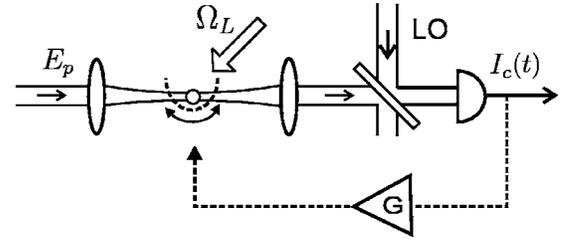


FIG. 2. Schematic setup for the continuous observation of the atomic momentum. The trapped atom is driven by a strong laser to create the transparency effect for the probe field  $E_p$ . Modulations of the probe light are detected in a homodyne measurement, i.e., by mixing  $E_p$  with the strong field of the local oscillator (LO). For feedback cooling, a force proportional to the measured signal  $I_c(t)$  is applied on the atom. Alternatively to the lens system, the atom can be placed inside a running wave cavity, e.g., to enhance the photon collection efficiency  $\epsilon$ .

probe field (“slow light”) while absorption is strongly suppressed (dotted line), giving rise to “electromagnetically induced transparency.” The width of the transparency window as well as the variation of the refractive index depend on  $\Omega_L$  and can be controlled by the laser field. We note the suppression of absorption at the dark state resonance [dotted line in Fig. 1(b)].

Consider now a single ion in a  $\Lambda$  configuration [Fig. 2(a)] moving in a trapping potential. We will only consider the one-dimensional (1D) motion along the propagation direction of both the probe and dressing laser beams. If we adopt for the moment a classical description of the ion motion with  $z(t)$  the ion trajectory, then the internal dynamics of the ion can again be described by the Hamiltonian (1) with a Doppler-shifted probe detuning

$$\Delta_p(t) = \Delta_p + (k_p - k_L)v(t),$$

where  $k_p$  and  $k_L$  the wave vectors of the running probe field and the dressing laser field, respectively, and  $v(t) \equiv \dot{z}(t)$  is the ion velocity. Thus for a resonant probe field,  $\Delta_p=0$ , the change in the index of refraction is a linear function of the atomic velocity or momentum with the steep slope given in Eq. (3).

This suggests the strategy of measuring the momentum of the ion continuously by monitoring the phase shift due to the varying index of refraction. We note that—while the index of refraction of a single particle is small—EIT will strongly amplify the sensitivity to the velocity. At the same time dissipation due to light scattering is strongly suppressed within the transparency window. These arguments can be easily adapted to a situation where the atomic motion is quantized.

## B. Continuous observation of the momentum of the trapped ion

We turn now to a formulation of the continuous read out of the ion momentum as outlined in Fig. 2. The idea is to measure the momentum of the ion via the phase changes of the probe beam as described in the previous sections. The

phase of the probe beam can be determined by homodyning, i.e., by mixing the probe beam with a local oscillator and measuring the homodyne current  $I_c(t)$ .

The state of the observed system (the moving ion) is described by a conditional density operator  $\mu_c(t)$ , which represents the observer's knowledge of the current state of the system for a given record of the measured signal,  $I_c(t)$ . We will show in Sec. III that after adiabatic elimination of the excited state of the (weakly driven) ion the measured homodyne current has the form

$$I_c(t) = 2\epsilon\Gamma_0\langle\hat{p}\rangle_c + \sqrt{\epsilon}\Gamma_0\xi(t). \quad (4)$$

The current is the sum of two terms. The first contribution shows a linear dependence on the conditional expectation value of the momentum operator  $\langle\hat{p}\rangle_c \equiv \text{Tr}\{\mu_c\hat{p}\}$ . Thus by measuring  $I_c(t)$  we learn the momentum of the moving ion. The second contribution describes a shot noise term with  $\xi(t)$ , a white noise Gaussian process. The signal strength is determined by the rate  $\Gamma_0$ . It is related to the slope of the refractive index  $\chi$  at  $\Delta_p=0$ : an explicit expression is given in Sec. III, Eq. (39) below. The parameter  $0 < \epsilon \leq 1$  takes into account the collection efficiency of the scattered photons (where in the ideal case  $\epsilon=1$ ). Equation (4) is derived under the assumption that  $\Omega_L$  is large compared to typical Doppler detuning,  $\Delta_D$ , and is valid on time scale which is slow compared to  $\Omega_L^{-1}$ . The signal is maximized for the local oscillator phase,  $\phi=0$ .

According to continuous measurement theory applied to homodyne detection, the conditional density operator  $\mu_c(t)$  is updated upon observation of the current  $I_c(t)$  following the Ito equation:

$$d\mu_c(t) = -i\nu[\hat{a}^\dagger\hat{a}, \mu_c(t)]dt + \mathcal{L}_M\mu_c(t)dt + \sqrt{\epsilon}\Gamma_0\mathcal{H}[\hat{p}]\mu_c(t)dW(t). \quad (5)$$

This equation will be derived in Sec. III, Eq. (38).

The first term in this equation describes the free evolution in the 1D harmonic trap where  $\nu$  denotes the trap frequency, and  $\hat{a}$  ( $\hat{a}^\dagger$ ) the destruction (creation) operator, respectively. The effects of the continuous observation appear in the second and third terms of Eq. (5).

The superoperator  $\mathcal{L}_M$  determines the back action of the measurement setup on the atomic motion. In the Lamb-Dicke limit  $\eta=2\pi a_0/\lambda_p \ll 1$ , where the extension of the atomic wavepacket (size of the harmonic oscillator ground state  $a_0$ ) is much smaller than the wavelength of the light,  $\lambda_p$ , it has the form,

$$\mathcal{L}_M\mu = -\frac{\Gamma_0}{2}[\hat{p}, [\hat{p}, \mu]]. \quad (6)$$

The action of  $\mathcal{L}_M$  tends to diagonalize the density operator in the eigenbasis of the (measured) operator  $\hat{p}$ . By comparing the decoherence rate  $\Gamma_0$  with the signal strength  $\epsilon\Gamma_0$  we see that for  $\epsilon < 1$  the measurement is not quantum limited, i.e., more noise is added than required by quantum mechanics [22]. Although the measurement does not reach the quantum limit, the back action is still minimal for a given collection efficiency.

In the third term of Eq. (5) we introduced the notation,

$$\mathcal{H}[\hat{c}]\mu = \hat{c}\mu + \mu\hat{c} - \langle c + \hat{c} \rangle \mu.$$

This term describes the observer's knowledge of the current state of the system and therefore depends on the measured signal. The Wiener increment  $dW(t)$  is formally related to the signal noise by  $dW(t) \equiv \xi(t)dt$ .

In summary, Eq. (4) for the homodyne current  $I_c(t)$  and the evolution equation (5) for the conditional density matrix constitute the basic equations of continuous observation of the momentum of the ion via homodyne detection.

### C. Quantum feedback cooling

The information on the atomic momentum contained in the signal  $I_c(t)$  (4) can be used to act back on the system. Here we are interested in cooling the atomic motion by using the feedback strategy known as ‘‘cold damping’’ [23]. The idea is to apply a force on the atom which is proportional but opposite to its momentum. This force creates an effective friction for the atomic motion and, therefore, leads to a dissipation of kinetic energy.

In our setup the measured signal is already proportional to the average momentum,  $\langle\hat{p}\rangle_c$ , and can be amplified and fed back directly. Thus we consider a feedback Hamiltonian of the form

$$H_{fb}(t) = \frac{G}{2\epsilon}I_c(t - \tau)\hat{z}, \quad (7)$$

where  $G$  is the dimensionless gain factor, and  $\tau$  is the finite delay in the feedback loop. Note that  $\tau > 0$ , so that  $H_{fb}$  acts after the measurement. Equation (5) with the feedback Hamiltonian added provides us with a *feedback equation* describing the time evolution of the system. The goal is now to average this equation over the Gaussian white noise  $\xi(t)$ .

A general theory for direct quantum feedback has been first discussed in a seminal paper by Wiseman and Milburn, to the best of our knowledge, [1]. In particular, they have shown how to average the quantum feedback equation in the limit  $\tau \rightarrow 0^+$ . In our case this assumption implies that the time delay of the feedback is small on the scale of the (adiabatically eliminated) system evolution, a condition which is realistic in the present context. Adopting this formalism we will derive in Sec. IV a master equation for the unconditional density operator,  $\mu(t) = E[\mu_c(t)]$ . We obtain (for  $\Delta_p = \Delta_L = 0$ )

$$\begin{aligned} \dot{\mu} = & -i\nu[\hat{a}^\dagger\hat{a}, \mu] + \mathcal{L}_M\mu - i\Gamma_0\frac{G}{2}[\hat{z}, \hat{p}\mu + \mu\hat{p}] \\ & - \Gamma_0\frac{G^2}{8\epsilon}[\hat{z}, [\hat{z}, \mu]]. \end{aligned} \quad (8)$$

The first line of this equation describes the free evolution and the measurement back action,  $\mathcal{L}_M$ , Eq. (6). The terms in the second line of Eq. (8) include the effects of the feedback loop. While the term proportional to  $G$  causes the expected damping of the motion, the second term leads to a diffusion of the atomic momentum. This diffusion originates from the

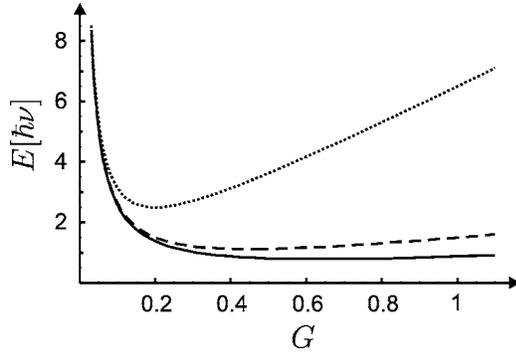


FIG. 3. The figure shows the dependence of the steady state energy as a function of the feedback gain,  $G$ . The results are calculated in the Lamb-Dicke limit and for  $\Gamma_0=0.01 \nu$ . The three curves are plotted for the parameters,  $\epsilon=0.1$  (solid line),  $\epsilon=0.05$  (dashed line), and  $\epsilon=0.01$  (dotted line).

noise in the measured current which is also amplified and fed back to the system.

The generic dependence of the steady state energy on the feed back gain calculated from Eq. (8) is plotted in Fig. 3. The curves show the expected minimum as a function of  $G$ , at the point where the noise of the feedback loop starts to dominate over the damping force. More detailed results will be presented in the following section.

#### D. Results: Quantum feedback vs EIT laser cooling

In general, the interaction of atoms with light always leads to some form of laser cooling or heating. In the resonant case  $\Delta_p=\Delta_L=0$  (discussed above), which is required to measure the atomic momentum, heating and cooling rates are equal and cause the diffusion described by  $\mathcal{L}_M$ .

By detuning the lasers away from the resonance,  $\Delta_L \neq 0$ , the atomic susceptibility,  $\chi(\Delta_p)$ , and therefore the absorption properties become quite asymmetric (see Fig. 4). This asymmetry is exploited in EIT laser cooling (ELC), where the absorption on the red sideband (a phonon is removed from the motion) is much more likely than on the blue sideband (a phonon is added to the motion). A detailed discussion of ELC can be found in Ref. [18].

In Sec. V we show that in the Lamb-Dicke limit we can derive the extension of Eq. (8) which includes quantum feedback cooling as well as ELC. In a frame rotating with the

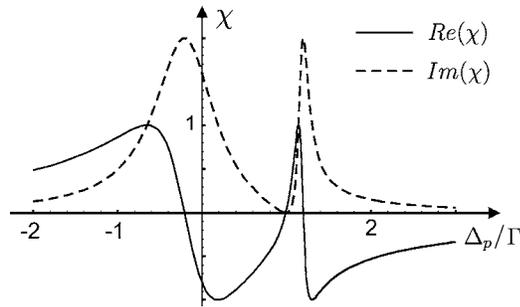


FIG. 4. Linear susceptibility  $\chi(\Delta_p)$  in arbitrary units for  $\Delta_L/\Gamma=\Omega_L/\Gamma=1$ .

trap frequency  $\nu$  it can be written in the form

$$\dot{\mu} = (A_- + A_-^{fb})\mathcal{D}[\hat{a}]\mu + (A_+ + A_+^{fb})\mathcal{D}[\hat{a}^\dagger]\mu, \quad (9)$$

with

$$\mathcal{D}[\hat{a}]\mu = \hat{a}\mu\hat{a}^\dagger - \frac{1}{2}\hat{a}^\dagger\hat{a}\mu - \frac{1}{2}\mu\hat{a}^\dagger\hat{a}.$$

The total cooling and the total heating rate are divided into a contribution from the atom-laser interaction,  $A_\pm$ , and a contribution from the feedback force,  $A_\pm^{fb}$ . They are given by

$$A_\pm = \frac{\Gamma_0}{2}\text{Re}[I(\pm\nu)],$$

$$A_\pm^{fb} = \Gamma_0 \left( G \frac{\nu\Gamma}{\Omega^2} \text{Im}[I^*(\pm\nu)e^{i\phi}] + \frac{G^2}{8\epsilon} \right).$$

The relation between the four rates is determined by the function  $I(\nu)$  which is defined as

$$I(\nu) = \frac{\Omega^4}{2\Gamma\nu^2} \frac{i\nu}{[\Omega^2 - 4\nu(\nu - \Delta_L)] + i2\Gamma\nu}. \quad (10)$$

Note that the rates  $A_\pm$  are proportional to the imaginary part of the atomic susceptibility at the sideband frequencies,  $\chi(\omega_p \mp \nu)$  (see Fig. 4). For a general detuning  $\Delta_L$ , the measured signal is no longer proportional to  $\langle \hat{p} \rangle$ , and therefore the phase of the local oscillator,  $\phi$ , appears in the feedback rates.

In the basis of harmonic oscillator states the master equation (9) has the form of a standard rate equation for the trap occupations ( $p_n = \langle n|\rho|n \rangle$ ) familiar from laser cooling,

$$\dot{p}_n = (A_- + A_-^{fb})[(n+1)p_{n+1} - np_n] + (A_+ + A_+^{fb})[(n-1)p_{n-1} - (n+1)p_n], \quad (11)$$

which predicts in a steady state a Bose-Einstein distribution with a mean occupation number

$$\bar{n} = \frac{A_+ + A_+^{fb}}{(A_- - A_+) + (A_-^{fb} - A_+^{fb})}. \quad (12)$$

We now turn to the discussion of results for the case of pure feedback cooling, and combined feedback and laser cooling:

*Pure feedback cooling.* We first reproduce the results of Eq. (8) by setting  $\Delta_p=\Delta_L=0$ . Then  $A_\pm=\Gamma_0/2$  and the cooling of the atom is attributed to the feedback mechanism. The minimal energy is reached for  $G=\sqrt{4\epsilon}$  and  $\phi=0$ . It is given by

$$E_{\min} = \frac{\hbar\nu}{2} \sqrt{\frac{1}{\epsilon}}. \quad (13)$$

This expression shows that the final temperature is only limited by the collection efficiency  $\epsilon$ . In the theoretical limit,  $\epsilon \rightarrow 1$ , it approaches the ground state energy,  $\hbar\nu/2$ .

*Feedback cooling and ELC.* When we tune away from the resonance  $\Delta_p=\Delta_L \neq 0$  we obtain a difference in the laser cooling rates  $A_- \neq A_+$ . In Fig. 5 we compare the final temperatures for the optimal feedback gain with the case of pure

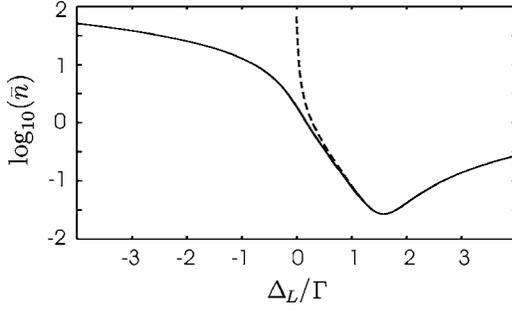


FIG. 5. Steady state occupation number,  $\bar{n} = \langle \hat{a}^\dagger \hat{a} \rangle$ , as a function of the detuning  $\Delta_L$ . The solid line shows the result for optimized  $\phi$  and optimized gain,  $G$ , while the dashed line shows the result without the feedback loop. The curves are plotted for the parameters:  $\epsilon = 0.05$ ,  $\Omega_L/\Gamma = 0.8$ , and  $\nu/\Gamma = 0.1$ .

ELC. For blue detuning,  $\Delta_L > 0$  the mechanism of ELC sets in and cools the atom close to the ground state. Although the addition of the feedback loop always leads to even lower temperatures, its effect can be neglected because for the present parameters ELC already provides efficient ground state cooling. For red detuning  $\Delta_L \leq 0$  the absorption spectrum (see Fig. 4) is reversed and the atom is actively heated by the light absorption. In this case a steady state is only reached when the feedback cooling dominates over the laser-induced heating.

### III. THE MODEL

In this section we present a detailed description of our model for the measurement setup which is shown in Fig. 2. The system of interest is the three-level atom which is confined by an external trapping potential. This atom is illuminated by a strong laser field to create the transparency effect for the probe field. The outgoing probe light is mixed with a strong local oscillator to perform a homodyne measurement to detect linear shifts of the field.

To describe the dynamics of the atom as well as the detection of scattered fields, we start with the total Hamiltonian,

$$H = H_A + H_{A-EM} + H_{EM}. \quad (14)$$

It is the sum of the Hamiltonian for the external and internal states of the atom,  $H_A$ , the free Hamiltonian for the electromagnetic environment,  $H_{EM}$ , and the coupling between the atom and the electromagnetic field,  $H_{A-E}$ .

For the internal level structure we consider a  $\Lambda$  configuration as shown in Fig. 1(a). A classical laser field with frequency  $\omega_L$  drives the transition between the excited state  $|e\rangle$  and the second ground or metastable state  $|r\rangle$ . The Rabi frequency for this coupling is denoted by  $\Omega_L$ . For the external dynamics of the atom we restrict ourselves to a one-dimensional model, i.e., we assume that the atom is strongly confined in the  $x$  and  $y$  directions. Along the  $z$  axis, which coincides with the propagation direction of the probe beam, the atom is trapped by the external potential  $V(z)$ . Although it is not essential for the following discussion, we further assume that  $V(z)$  is harmonic, with a trap frequency  $\nu$ .

This assumption allows us to introduce the dimensionless position and momentum operators,  $\hat{z} := (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$  and  $\hat{p} := i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$ , where  $\hat{a}$  and  $\hat{a}^\dagger$  denote the usual annihilation and creation operators. With the definition of the external Hamiltonian,  $H_E = \hbar \nu \hat{a}^\dagger \hat{a}$ , and the notation  $\sigma_{ij} = |i\rangle\langle j|$  the atom Hamiltonian is then given by

$$H_A = H_E + \hbar \omega_e |e\rangle\langle e| + \hbar \omega_r |r\rangle\langle r| + \frac{\hbar \Omega_L}{2} (e^{-i\omega_e t} e^{i\eta_L \hat{z}} \sigma_{er} + \text{H.c.}). \quad (15)$$

Here  $\omega_{e,r}$  denote the eigenfrequencies of the corresponding states  $|e\rangle$  and  $|r\rangle$ . The Lamb-Dicke parameter is defined as,  $\eta_r = k_L \sqrt{\hbar/m\nu}$ , where  $m$  is the mass of the atom and  $k_L$  is the wave vector (projected on the  $z$  axis) of the laser field.

The electromagnetic environment consist of a three-dimensional set of plane wave modes, labeled by their wave vector  $\vec{k}$  and their polarization  $\lambda$ . In terms of the corresponding annihilation and creation operators  $\hat{b}_\lambda(\vec{k})$  and  $\hat{b}_\lambda^\dagger(\vec{k})$ , the free evolution is determined by the Hamiltonian

$$H_{EM} = \sum_{\lambda=1,2} \int d^3k \hbar \omega_k \hat{b}_\lambda^\dagger(\vec{k}) \hat{b}_\lambda(\vec{k}). \quad (16)$$

The electric field of the environment interacts with the internal states of the atom via a dipole coupling. Under the rotating wave approximation the interaction Hamiltonian is

$$H_{A-E} = -\vec{\mu}_{eg} \vec{E}^+(\hat{z}) \sigma_{eg} - \vec{\mu}_{er} \vec{E}^+(\hat{z}) \sigma_{er} + \text{H.c.}, \quad (17)$$

with the standard expression for the electric field

$$\vec{E}^+(\vec{x}) = i \sum_{\lambda=1,2} \int d^3k \sqrt{\frac{\hbar \omega_k}{2\epsilon_0(2\pi)^3}} \vec{\epsilon}_\lambda(\vec{k}) \hat{b}_\lambda(\vec{k}) e^{i\vec{k}\vec{x}}. \quad (18)$$

In an experiment the lens system defines a certain spatial mode function for the probe beam. To describe the homodyne detection of the probe field, we divide the total electric field into the field of this particular mode  $\vec{E}_p$  and a remaining set of modes, orthogonal to  $\vec{E}_p$ ,

$$\vec{E}^+(\vec{x}) = \vec{E}_p^+(\vec{x}) + \vec{E}_\perp^+(\vec{x}). \quad (19)$$

In our model we approximate  $\vec{E}_p$  by the one-dimensional field,

$$\vec{E}_p^+(z) = i \vec{\epsilon}_p e^{i(k_p z - \omega_p t)} + i \vec{\epsilon}_p \int_0^\infty dk a(k) e^{ikz} \hat{b}_p(k). \quad (20)$$

The first part in this expression describes the coherent field of the incoming probe beam. Note that with this definition of the operator  $\vec{E}_p$ , the initial state of the electromagnetic environment is the vacuum state. The function  $a(k)$  determines the coupling of the atom to the dense set of modes  $\hat{b}_p(k)$ . It must be adjusted to reproduce the correct results of the real mode function.

The interaction of the atom with the coherent part of the probe field leads to transitions between  $|g\rangle$  and  $|e\rangle$ , characterized by the Rabi frequency  $g = 2|\vec{\mu}_{eg} \vec{\epsilon}_p|/\hbar$ . In the following we include this term in the atomic evolution and define a

new system Hamiltonian,  $H_S = H_A + H_{A-EM}|_{\text{coh}}$ . In a frame rotating with the laser frequencies  $\omega_p$  and  $\omega_L$ , this Hamiltonian is then given by

$$H_S = H_E - \hbar\Delta_p|e\rangle\langle e| - \hbar(\Delta_p - \Delta_L)|r\rangle\langle r| + \frac{\hbar\Omega_L}{2}(e^{i\eta_r\hat{z}}\sigma_{er} + e^{-i\eta_r\hat{z}}\sigma_{re}) + \frac{\hbar g}{2}(e^{i\eta_g\hat{z}}\sigma_{eg} + e^{-i\eta_g\hat{z}}\sigma_{ge}). \quad (21)$$

Here we introduced the detunings  $\Delta_p = \omega_p - \omega_e$  and  $\Delta_L = \omega_L - (\omega_e - \omega_r)$ , and the Lamb-Dicke parameter of the probe field,  $\eta_g = k_p \sqrt{\hbar/mv}$ .

The coherent evolution of the driven atom is now determined by the system Hamiltonian  $H_S$ . The coupling between the atom and the dense set of modes of  $\vec{E}_\perp$  and the nonclassical part of  $\vec{E}_p$  has two effects. First, it leads to an incoherent dynamic of the atomic state. This includes the decay of the excited state population as well as a diffusion of the atomic momentum due to the random recoil kicks of the emitted photons. Second, the coupling changes the state of the electromagnetic environment which is (partially) detectable by the observer.

### A. Master equation

We first look at the incoherent dynamics of the atom and ignore the evolution of the electromagnetic bath. By applying the standard Born-Markov approximation and tracing over the bath degrees of freedom we obtain a master equation for the system density matrix,  $\rho$ . It can be written in the standard form,  $\dot{\rho} = \mathcal{L}\rho$ , with a Liouville operator [18,24,25],

$$\mathcal{L}\rho = -\frac{i}{\hbar}[H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger] + \mathcal{J}_g(\sigma_{ge}\rho\sigma_{eg}) + \mathcal{J}_r(\sigma_{re}\rho\sigma_{er}). \quad (22)$$

The effective Hamiltonian,  $H_{\text{eff}} = H_S - i\hbar\Gamma/2|e\rangle\langle e|$ , includes the unitary evolution of the atom as well as the decay of the excited state population with a total rate  $\Gamma$ . The two ‘‘recycling’’ terms are defined by

$$\mathcal{J}_{j=r,g}(\rho) = \Gamma_j \int_{-1}^1 du N(u) e^{-i\eta_j \hat{z} u} \rho e^{i\eta_j \hat{z} u}. \quad (23)$$

The rates  $\Gamma_g$  and  $\Gamma_r$  denote the decay rates into the corresponding states,  $|g\rangle$  and  $|r\rangle$ . The dipole distribution  $N(u) = \frac{3}{8}(1+u^2)$ , with  $u = \cos(\varphi)$ , determines the probability for emitting a photon under a certain angle  $\varphi$  with respect to the  $z$  axis.

Master equation (22) describes the full dynamics of a driven atom in a  $\Lambda$  configuration. The external and internal degrees of freedom are coupled via the position-dependent interaction with the electromagnetic field. In general, the recoil kicks of the emitted photons described by  $\mathcal{J}_{g,r}$  lead to momentum diffusion and a heating of the atomic motion. For an appropriate choice of laser detunings the heating can be compensated by photon absorptions (laser cooling). In this paper we follow a different approach where cooling is provided by an external feedback force.

### B. Continuous homodyne detection

In a next step we describe the homodyne detection of the probe field  $\vec{E}_p$ . Here we follow the standard theory of homodyne detection (see, e.g., Refs. [26–28]) to derive the relevant equations for the model specified above.

After the interaction with the atom, the outgoing probe field is described by the Heisenberg operator  $\hat{b}_{p,\text{out}}(t)$ . Using the input-output formalism [26] it is related to the incoming field,  $\hat{b}_{p,\text{in}}(t)$ , by

$$\hat{b}_{p,\text{out}}(t) = \hat{b}_{p,\text{in}}(t) + \sqrt{\gamma}\hat{c}_p(t), \quad (24)$$

where  $\hat{c}_p = e^{-i\eta_p \hat{z}}\sigma_{ge}$  denotes the atomic ‘‘jump operator’’ which couples to the probe field. To relate our model to the real experimental setup, we set  $\gamma = \epsilon\Gamma_g$ , where the collection efficiency  $\epsilon$  determines the fraction of photons which are scattered into the mode of the probe field.

In the homodyne detection the outgoing field (24) is mixed with a strong coherent field of the local oscillator. When the transmittance of the beam splitter is close to one the field operator at the position of the detector is

$$\hat{b}_d(t) = \sqrt{\gamma}\beta e^{i\phi} e^{-i\omega_p t} + \hat{b}_{p,\text{in}}(t) + \sqrt{\gamma}\hat{c}_p(t). \quad (25)$$

Here,  $\beta$  and  $\phi$  denote the real amplitude and the phase of the reflected part of the local oscillator. Note that the total probe field defined in Eq. (20) is the sum of a classical and a quantized contribution. The classical part of  $\hat{b}_{p,\text{in}}(t)$  can simply be absorbed in a redefinition of  $\beta e^{i\phi}$ . The operator for the homodyne current is

$$\hat{I}_h(t) = \lim_{\beta \rightarrow \infty} (\hat{b}_d^\dagger(t)\hat{b}_d(t) - \gamma\beta^2)/\beta. \quad (26)$$

The measured signal  $I_c(t)$  is then defined as the outcome of the continuous measurement of the current operator  $\hat{I}_h(t)$ . Using the results from the theory of homodyne detection [26], we obtain

$$I_c(t) = \epsilon\Gamma_g \langle \hat{c}_p e^{-i\phi} + \hat{c}_p^\dagger e^{i\phi} \rangle_c(t) + \sqrt{\epsilon\Gamma_g} \xi(t). \quad (27)$$

The unconditioned evolution given by master equation (22) and the measurement record,  $I_c(t)$ , determine the evolution of the conditioned density operator,  $\rho_c(t)$ . Following Ref. [28] we obtain

$$d\rho_c(t) = \mathcal{L}\rho_c(t)dt + \sqrt{\epsilon\Gamma_g} \mathcal{H}[\hat{c}_p e^{-i\phi}] \rho_c(t) dW(t). \quad (28)$$

Equations (27) and (28) represent a full description of the conditioned dynamics of the atom under continuous observation and serve as the starting point for the following discussion.

### C. Adiabatic elimination

The current  $I_c(t)$  as given in Eq. (27) is still a function of the coupled external and internal states of the atom. In the following we show that for a weakly excited atom we can eliminate the dynamics of the internal states, and the measured signal becomes a linear function of the atomic momentum as given in Eq. (4). In addition we derive the resulting

back action of the measurement on the motional state of the atom.

As already noted in Sec. II, the phase shift of the probe light is a linear function of the atomic momentum as long as the typical Doppler detuning  $\Delta_D = \nu \eta_p \langle \hat{p} \rangle$  is small compared to the width of the transparency window  $\Omega_L$ . Therefore, we can apply perturbation theory in the parameter  $\Delta_D/\Omega_L$  to derive an effective equation for the external state. As a first step in our calculation we perform a unitary transformation,

$$U = e^{i\eta_p \hat{z}|e\rangle\langle e|} e^{-i\bar{\eta} \hat{z}|r\rangle\langle r|}, \quad (29)$$

where we set  $\bar{\eta} := \eta_p - \eta_L$ . In the new basis the system Hamiltonian  $\tilde{H}_S = U^\dagger H_S U$  is given by

$$\begin{aligned} \tilde{H}_S = & H_E - \hbar(\Delta_p - \nu \eta_p \hat{p} - \nu \eta_p^2/2)|e\rangle\langle e| - \hbar(\Delta_p - \Delta_L + \nu \bar{\eta} \hat{p} \\ & - \nu \bar{\eta}^2/2)|r\rangle\langle r| + \frac{\hbar \Omega_L}{2}(\sigma_{er} + \sigma_{re}) + \frac{\hbar g}{2}(\sigma_{eg} + \sigma_{ge}). \end{aligned} \quad (30)$$

As in the classical case (see Sec. II A) the position dependence of the atom-laser coupling is transformed into a frequency shift for the states  $|e\rangle$  and  $|r\rangle$ . In addition to the Doppler detunings  $\nu \eta_p \hat{p}$  and  $\nu \bar{\eta} \hat{p}$ , the internal states are also shifted by the appropriate recoil frequencies. They account for the fact, that each absorption or emission of a photon also transfers kinetic energy to the atom.

For the particular choice of laser detunings  $\Delta_p = \Delta_L + \nu \bar{\eta}^2/2$ , and in the absence of a trapping potential the Hamiltonian  $\tilde{H}_S$  has a dark eigenstate,  $|\psi\rangle_D = |p=0, D\rangle$ . Here  $|p=0\rangle$  is the zero momentum eigenstate and  $|D\rangle$  denotes the internal dark state,

$$|D\rangle = (|g\rangle_r - \Omega_L |g\rangle_g)/\Omega, \quad (31)$$

with  $\Omega = \sqrt{\Omega_L^2 + g^2}$ . In the following we assume that this relation between the detunings is fulfilled. The system Hamiltonian can then be written as

$$\tilde{H}_S = H_E + H_I + H_{\text{int}}, \quad (32)$$

such that  $H_E$  and  $H_I$  act on the external or the internal states only, while  $H_{\text{int}}$  describes the coupling between them;

$$H_{\text{int}} = \hbar \nu \eta_p \hat{p} |e\rangle\langle e| - \hbar \nu \bar{\eta} \hat{p} |r\rangle\langle r|. \quad (33)$$

With the definition  $\Delta = \Delta_p - \nu \eta_p^2/2$  the Hamiltonian of the internal states  $H_I$  reduces to the one of a driven  $\Lambda$  system at the two photon resonance,

$$H_I = -\hbar \Delta |e\rangle\langle e| + \frac{\hbar \Omega_L}{2}(\sigma_{er} + \sigma_{re}) + \frac{\hbar g}{2}(\sigma_{eg} + \sigma_{ge}). \quad (34)$$

In the limit where the external and internal degrees of freedom decouple,  $H_{\text{int}} \rightarrow 0$ , the conditioned dynamics determined by Eq. (28) leads to a relaxation of the atom into the state

$$\tilde{\rho}_c(t \rightarrow \infty) = \tilde{\mu}_0 \otimes |D\rangle\langle D|, \quad (35)$$

where  $\tilde{\mu}_0$  is an undetermined state of the external degrees of freedom. The interaction with the atomic motion,  $H_{\text{int}}$ , or to be more precise the term  $\hbar \nu \bar{\eta} \hat{p} |r\rangle\langle r|$  couples the state  $|D\rangle$  to the bright (internal) eigenstates of  $H_I$ ,  $|+\rangle$  and  $|-\rangle$ . They are given by

$$|+\rangle = \cos(\theta)|e\rangle + \sin(\theta)(|g\rangle_g + \Omega_L |r\rangle)/\Omega,$$

$$|-\rangle = \sin(\theta)|e\rangle - \cos(\theta)(|g\rangle_g + \Omega_L |r\rangle)/\Omega,$$

where the mixing angle  $\theta$  is defined by the relation  $\tan(\theta) = \Omega/(\sqrt{\Omega^2 + \Delta^2} - \Delta)$ . These two states are separated from the dark state by the energies

$$\hbar \Omega_{\pm} = -\hbar(\Delta \mp \sqrt{\Omega^2 + \Delta^2})/2. \quad (36)$$

Therefore, for small Doppler shifts the population in the bright states is of the order of  $\Delta_D^2/\Omega_{\pm}^2$ .

In the following we consider the limit where the eigenfrequencies of the bright states,  $\Omega_{\pm}$ , are much larger than the typical Doppler detuning  $\Delta_D$ , as well as the frequency of the trap,  $\nu$ . The first assumption says that the atom is only weakly excited and the total density operator is well approximated by

$$\tilde{\rho}_c(t) \approx \tilde{\mu}_0(t) \otimes |D\rangle\langle D|. \quad (37)$$

The second condition  $\Omega_L \gg \nu$  ensures us that the internal state of the atom adiabatically follows the evolution of the atomic momentum. If both conditions are satisfied we can adiabatically eliminate the population in the bright states and derive an effective equation for the evolution of the motional state,  $\tilde{\mu}_0(t)$ . The details of this calculation are summarized in Appendix A. Finally, we revert the unitary transformation  $U$  (29), and trace over the internal states. For the resulting conditioned density operator of the motional state,  $\mu_c := \text{Tr}_I\{U \tilde{\rho}_c U^\dagger\}$ , we obtain the stochastic master equation,

$$\begin{aligned} d\mu_c(t) = & -i[\nu \hat{a}^\dagger \hat{a} - \lambda^2 \Delta \hat{p}^2, \mu_c(t)]dt + \mathcal{L}_M \mu_c(t)dt \\ & + \sqrt{\epsilon \lambda^2 \Gamma} \mathcal{H}[\hat{p} e^{-i\phi}] \mu_c(t) dW(t), \end{aligned} \quad (38)$$

and the homodyne current

$$I_c(t) = 2\epsilon \lambda^2 \Gamma_g \cos(\phi) \langle \hat{p} \rangle_c + \sqrt{\epsilon \lambda^2 \Gamma} g \xi(t). \quad (39)$$

In these two equations we defined the parameter,  $\lambda = 2g \bar{\eta} \nu / \Omega^2$ , and the average is taken with respect to the conditioned motional state,  $\langle \cdot \rangle_c = \text{Tr}\{\mu_c \cdot\}$ . The measurement back action has the form

$$\mathcal{L}_M \mu = \frac{\Gamma_0}{2} \left[ \frac{2\tilde{\mathcal{J}}_g}{1 - \tilde{\mathcal{J}}_r} (\hat{p} \mu \hat{p}) - \hat{p}^2 \mu - \mu \hat{p}^2 \right], \quad (40)$$

with  $\Gamma_0 = \lambda^2 \Gamma$  and

$$\tilde{\mathcal{J}}_{j=g,r}(\mu) = \frac{\Gamma_j}{\Gamma} \int_{-1}^1 du N(u) e^{-i\eta_j \hat{z}(u-1)} \mu e^{i\eta_j \hat{z}(u-1)}. \quad (41)$$

*Discussion.* The results given in Eqs. (38) and (39) are valid in the limit of a weak probe field  $g \ll \Omega_L$ . In that case

the dark state  $|D\rangle$  almost coincides with the ground state  $|g\rangle$ , and the signal strength is maximized for a given strength of the measurement back action  $\Gamma_0$ , (see Appendix A). The signal  $I_c(t)$  can further be optimized setting  $\phi=0$  and by choosing atomic states with a small branching ratio  $\Gamma_r/\Gamma_g$ . The remaining difference between  $\lambda^2\Gamma_g$  and  $\Gamma_0$  can be absorbed into the definition of  $\epsilon$ . For  $\Delta=0$  we then end up with Eqs. (4) and (5) as given in Sec. II B.

#### IV. FEEDBACK COOLING

As already mentioned in Sec. II the goal of the continuous momentum observation is to use the information in the signal to manipulate the motion of the atom, e.g., to cool it. In this section we discuss the implementation of the cold damping feedback strategy. By applying the theory of direct quantum feedback [1] we derive a master equation for the unconditioned state,  $\mu(t)=E[\mu_c(t)]$ .

For the feedback cooling we consider the measurement setup as described in the previous section. In addition, we apply a force on the atom which is proportional but opposite to the measured current. For a single trapped ion, such a feedback loop can be realized by converting the homodyne current into a voltage difference between two trap electrodes. The effect of the feedback loop on the system evolution can be written in the general form,

$$\dot{\mu}_c(t)|_{fb} = I_c(t - \tau)\mathcal{K}\mu_c(t). \quad (42)$$

The time delay of the feedback loop  $\tau$  can usually be neglected compared to the timescale of the atomic motion  $\nu^{-1}$ . Nevertheless, for the derivation of the final master equation a finite value of  $\tau$  is important to obtain the correct operator ordering [1]. To implement the idea of cold damping, we consider the feedback superoperator,

$$\mathcal{K}\mu = -i\frac{G}{2\epsilon}[\hat{z}, \mu], \quad (43)$$

where  $G$  denotes the dimensionless gain factor. Note that with this definition of  $\mathcal{K}$ , the frequency scale of the feedback contribution is again of order  $\Gamma_0$ .

It has been shown in Ref. [1] that Eq. (42) must be interpreted in the Stratonovich sense. To be compatible with Eq. (5) we convert it into the Ito-type equation,

$$d\mu_c(t)|_{fb} = \Gamma_0 \left( 2\epsilon\langle\hat{p}\rangle_c(t-\tau)\mathcal{K} + \frac{\epsilon}{2}\mathcal{K}^2 \right) \mu_c(t)dt + \sqrt{\epsilon\Gamma_0}\mathcal{K}\mu_c(t)dW(t-\tau). \quad (44)$$

In this form we already see that the noise in the current  $I_c(t)$  leads to the diffusion term  $\mathcal{K}^2$ . We can now add Eq. (44) to the conditioned evolution given in Eq. (5) and obtain the full stochastic dynamics of the atom under the action of the feedback loop.

To derive a master equation which is independent of the measurement outcome, we perform an ensemble average over the stochastic process,  $\xi(t)$  and obtain the evolution

of the unconditioned density operator,  $\mu=E[\mu_c]$ . When taking the average we must keep in mind that although  $E[dW(t)]=0$ , the Ito increment,  $dW(t-\tau)$ , is not independent of  $\mu_c(t)$ , e.g.,  $E[dW(t-\tau)\mu_c(t)]\neq 0$ . The way to perform the average in the limit,  $\tau\rightarrow 0^+$ , can be found in Ref. [29]. By following this procedure we end up with the master equation (8) given in Sec. II.

#### A. Feedback cooling in the Lamb-Dicke limit

We first look at the solution of Eq. (8) in the Lamb-Dicke limit,  $\eta_p, \eta_L \ll 1$ . In this limit the recoil kicks of the emitted photons can be neglected and the backaction of the measurement (40) simplifies to

$$\mathcal{L}_M\mu = -\frac{\Gamma_0}{2}[\hat{p}, [\hat{p}, \mu]]. \quad (45)$$

For a harmonic trapping potential  $H_E = \hbar\nu\hat{a}^\dagger\hat{a}$ , the feedback master equation (8) is then quadratic in the position and the momentum operators. Therefore, the final state is Gaussian and we obtain analytic expressions for the variances of  $\hat{z}$  and  $\hat{p}$ . The resulting steady state energy is given by

$$E = \frac{\hbar\nu}{2} \left( \frac{G\Gamma_0^2}{2\nu^2} + \frac{1}{G} + \frac{G}{4\epsilon} \right). \quad (46)$$

The first contribution in the brackets originates from the enhanced uncertainty of the position coordinate as a result of the measurement of the momentum operator. If the measurement strength  $\Gamma_0$  is much smaller than the trap frequency  $\nu$ , position and momentum coordinates are mixed sufficiently fast and this contribution disappears. In this limit the optimal feedback gain is given by  $G=\sqrt{4\epsilon}$  and we obtain the minimal energy given in Eq. (13).

In Sec. V we extend the discussion of the feedback cooling in the Lamb-Dicke limit to arbitrary detunings  $\Delta$ . Then laser cooling effects play an important role for the final temperatures.

#### B. Feedback cooling beyond the Lamb-Dicke limit

When the trapping potential is weak, the extension of the atomic wavepacket can be of the order of the wavelength of the emitted photons. In that case, the energy spacing in the trap  $\hbar\nu$  is comparable to the recoil energy  $E_R$ , and recoil kicks from the emitted photons lead to an additional diffusion of the atomic momentum. Therefore, we must take into account the full expression for the back action term  $\mathcal{L}_M$ . By expanding Eq. (40) in the parameter  $\Gamma_r/\Gamma$  we can write it as

$$\mathcal{L}_M\mu = \frac{\Gamma_0}{2} \left[ 2\tilde{\mathcal{J}}_g \left( \sum_{n=0}^{\infty} \tilde{\mathcal{J}}_r^n \right) (\hat{p}\mu\hat{p}) - \hat{p}^2\mu - \mu\hat{p}^2 \right]. \quad (47)$$

The zeroth order term in the sum corresponds to the physical picture where the atom is excited and simply decays back to the ground state,  $|g\rangle \approx |D\rangle$ . Processes where the atom first decays into the state  $|r\rangle$ , and is then reexcited are taken into account by including higher order terms in this sum.

In general, the full expression of  $\mathcal{L}_M$  leads to a hierarchy of coupled equations for the moments of  $\hat{p}$  which do not break off as in the Lamb-Dicke limit. In the following we restrict our discussion to a finite trapping potential and consider the limit  $\Gamma_0 \ll \nu$ . As mentioned above, this ensures a mixing of position and momentum coordinates and therefore, an equal reduction of both variances. In this regime nonenergy conserving terms can be neglected and we obtain an equation for the mean occupation number

$$\langle \dot{\hat{n}} \rangle = -\Gamma_0 [G - D] \langle \hat{n} \rangle + \frac{\Gamma_0}{2} \left[ \frac{G^2}{4\epsilon} - G + 1 + D \right]. \quad (48)$$

In this expression the parameter  $D$  describes the heating induced by the recoil kicks from the emitted photons. It is given by

$$D = \eta_g^2 \tilde{\alpha} + \frac{\Gamma_r}{\Gamma} \left[ \frac{\Gamma_g}{\Gamma_g - \Gamma_r} (\eta_r^2 \tilde{\alpha} + \eta_r \eta_g) + \frac{\Gamma_r}{\Gamma_g - \Gamma_r} \eta_r^2 \right], \quad (49)$$

where  $\tilde{\alpha} = \frac{1}{2} \int N(u) (u-1)^2 du = \frac{7}{10}$ . By choosing an appropriate atomic level configuration, i.e.,  $\Gamma_g \gg \Gamma_r$  and/or  $\eta_g^2 \gg \eta_r^2$ , its value is only limited by  $D \approx \eta_g^2 \tilde{\alpha}$ . In this case, and for optimized gain the minimal steady state energy is

$$E_{\min} = \hbar \nu \left( \frac{\eta_g^2 \tilde{\alpha} + \sqrt{4\epsilon + \eta_g^4 \tilde{\alpha}^2}}{4\epsilon} \right). \quad (50)$$

This expression shows that the minimal energy changes from the Lamb-Dicke to the non-Lamb-Dicke regime at the parameter values  $4\epsilon \approx \eta_g^2 \tilde{\alpha}$ . In the non-Lamb-Dicke regime, i.e., for weak trapping potential, the minimal energy approaches the value,

$$E_{\min} \approx \frac{\tilde{\alpha}}{\epsilon} E_R. \quad (51)$$

Therefore, the limit for feedback cooling is set by the recoil energy  $E_R$ , divided by  $\epsilon$  and temperatures well below the Doppler limit,  $k_B T_D = \hbar \Gamma / 2$ , can be reached.

## V. FEEDBACK VS EIT LASER COOLING

In the previous section we focused on the laser detunings  $\Delta_p \approx \Delta_L \approx 0$  with the goal to measure the atomic momentum to achieve quantum feedback cooling. As already discussed in Sec. II D, in a  $\Lambda$  systems EIT laser cooling provides an effective mechanism to cool atoms essentially to the ground state without any further external manipulation. In this section we derive a master equation which describes both effects, feedback cooling and ELC, and discuss the cross-over from pure feedback cooling to ELC.

For the adiabatic elimination of the excited states in Sec. III C and therefore for the validity of the feedback master equation (8) we required,  $\nu \ll \Omega_{\pm}$ . This assumption excludes the parameter regime, where ELC achieves the lowest temperatures,  $\nu \sim \Omega_{\pm}$  [18]. In this section we restrict the discussion to the Lamb-Dicke limit  $\eta_g, \eta_r \ll 1$ . This allows us to derive a master equation for the motional state for arbitrary choice of the parameters  $\Delta, \Omega_L$ , and  $\nu$ .

We start with the full model for the three-level atom coupled to the radiation field as introduced in Sec. III. To optimize the feedback cooling effect and to simplify the following discussion we make the assumptions  $\Gamma_g = \Gamma$  and as in the previous sections  $g \ll \Omega_L$ . Under the two photon resonance condition  $\Delta_p = \Delta_L = \Delta$ , the system Hamiltonian,  $H_S$ , given in Eq. (21), can be written as

$$H_S = H_E + H_I + H_{\eta}. \quad (52)$$

The Hamiltonian  $H_{\eta}$  describes the coupling between external and internal degrees of freedom. Up to first order in the Lamb-Dicke parameters it is given by

$$H_{\eta} \approx i \hbar \hat{z} \left[ \eta_r \frac{\Omega_L}{2} (\sigma_{er} - \sigma_{re}) + \eta_g \frac{g}{2} (\sigma_{eg} - \sigma_{ge}) \right]. \quad (53)$$

The conditioned dynamics of the full atomic density operator,  $\rho_c(t)$ , is determined by the stochastic master equation (28). As in Sec. III C the goal is to eliminate the internal states and to derive an effective equation for conditioned motional density operator,  $\mu_c(t)$ . The principal strategy is the same: For vanishing Lamb-Dicke parameters the decay of the bright states relaxes the atom into the state,  $\rho_c(t) = \mu_c(t) \otimes |D\rangle\langle D|$ . The dynamics of  $\mu_c(t)$  can be derived by including the coupling Hamiltonian  $H_{\eta}$  in second order perturbation theory. In contrast to Sec. III C we impose no restrictions on the energies of the bright states,  $\Omega_{\pm}$ , which lead to resonant transitions for  $|\Omega_{\pm}| \approx \nu$ . Therefore, to guarantee the validity of the perturbation theory we require that  $\langle H_{\eta} \rangle \approx \bar{\eta} g$  is much smaller than the decay rates of the bright states,  $\Gamma_{+} \sim \Gamma \cos^2(\theta)$  and  $\Gamma_{-} \sim \Gamma \sin^2(\theta)$ .

In Appendix B we use the stochastic Schrödinger equation formalism for the adiabatic elimination of the internal states. As a result we obtain the conditioned master equation,

$$d\mu_c = -i(\nu + \delta)[\hat{a}^{\dagger} \hat{a}, \mu_c] dt + A_- \mathcal{D}[\hat{a}] \mu_c dt + A_+ \mathcal{D}[\hat{a}^{\dagger}] \mu_c dt + \sqrt{\epsilon \Gamma_0} \mathcal{H}[\hat{C} e^{-i\phi}] \mu_c dW(t). \quad (54)$$

and the expression for the homodyne current,

$$I_c(t) = \epsilon \Gamma_0 \langle \hat{C} e^{-i\phi} + \hat{C}^{\dagger} e^{i\phi} \rangle_c(t) + \sqrt{\epsilon \Gamma_0} \xi(t). \quad (55)$$

Here we set  $\delta = \Gamma_0 \text{Im}[I(-\nu) + I(\nu)]/4$  and defined the atomic ‘‘jump operator,’’

$$\hat{C} = \frac{\sqrt{2\nu}\Gamma}{\Omega^2} [I(-\nu)\hat{a} + I(+\nu)\hat{a}^{\dagger}], \quad (56)$$

This operator as well as the laser heating and cooling rates,  $A_{\pm} = \Gamma_0 \text{Re}[I(\pm\nu)]/2$ , depend on the function  $I(\pm\nu)$ , which is defined in Sec. II, Eq. (10).

As in the previous section we consider a feedback force which is proportional to the measured signal,  $I_c(t)$ . Note that depending on values of  $I(\pm\nu)$ , and the local oscillator phase,  $\phi$ , the force is proportional to a linear combination of  $\langle \hat{p} \rangle$  and  $\langle \hat{z} \rangle$ . For the derivation of the feedback master equation we follow the outline given in Sec. IV and obtain

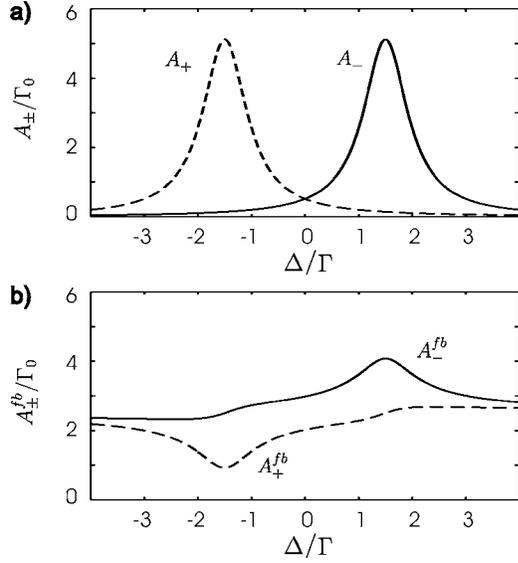


FIG. 6. (a) Laser heating and cooling rates,  $A_{\pm}$ , as a function of the laser detuning  $\Delta$ . (b) Feedback heating and cooling rates,  $A_{\pm}^{fb}$ , for  $G=1$  and an optimized local oscillator phase  $\phi$ . The parameters for the two plots are  $\epsilon=0.05$ ,  $\Omega/\Gamma=0.8$ , and  $\nu/\Gamma=0.1$ .

$$\begin{aligned} \dot{\mu} = & -i(\nu + \delta)[\hat{a}^\dagger \hat{a}, \mu] + A_- \mathcal{D}[\hat{a}] \mu + A_+ \mathcal{D}[a^\dagger] \mu \\ & - i\Gamma_0 \frac{G}{2} [\hat{z}, \hat{C} e^{-i\phi} \mu + \mu \hat{C}^\dagger e^{i\phi}] - \Gamma_0 \frac{G^2}{8\epsilon} [\hat{z}, [\hat{z}, \mu]]. \end{aligned} \quad (57)$$

Under the rotating wave approximation, which is valid for  $\Gamma_0 \ll \nu$ , and by neglecting small shifts of the trap frequency, we end up with master equation (9) given in Sec. II.

*Discussion.* Figure 6 shows the dependence of the four different rates  $A_{\pm}, A_{\pm}^{fb}$  as a function of the detuning  $\Delta$ . The cooling and heating rates which originate from the laser interaction,  $A_{\pm}$ , correspond to the rates for ELC derived in Ref. [18]. For the parameter regime  $\Omega > 2\nu$  and for blue detuning,  $\Delta > 0$ , they lead to a minimal temperature for  $\Omega^2 \approx 4\nu(\nu - \Delta)$ . For red detuning,  $\Delta \leq 0$ , the heating rate is larger than the cooling rate and without feedback the system does not reach a steady state. By adjusting the phase  $\phi$  the feedback loop always provides additional damping,  $W = A_-^{fb} - A_+^{fb} > 0$ , which for  $\Delta = 0$  and  $\nu \ll \Omega$  is given by  $W = G\Gamma_0$ . The noise added by the feedback loop  $\Gamma_0 G^2 / 8\epsilon$ , imposes a restriction on  $G$  if one is interested in low steady state energies. The combined effect of feedback and EIT laser cooling lead to a final temperature which is plotted in Fig. 3 in Sec. II D.

## VI. CONCLUSION

In this paper we have shown that a continuous readout of the momentum of a single atom can be achieved by employing the high velocity sensitivity of the index of refraction in a driven  $\Lambda$  system. The transparency effect for an atom at rest and the linear dependence of the index of refraction on the Doppler shift lead to a homodyne current linear in  $\langle \hat{p} \rangle$

and a minimal back action on the atomic motion, approaching the quantum limit for  $\epsilon \rightarrow 1$ .

By applying a force which is proportional to the measured signal, feedback cooling for single ions can be realized. The cooling scheme is applicable in and outside the Lamb-Dicke regime with steady state temperatures well below the Doppler limit. From a fundamental point of view we want to point out that, in the proposed feedback scheme, the measured current is fed back *directly* on the trap electrodes. Therefore, its implementation allows for a test of the theory of direct quantum feedback [1] on an individual quantum system close to the ground state.

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## APPENDIX A: ADIABATIC ELIMINATION

In this appendix we derive the stochastic master equation (38) for the external density operator  $\rho_c$  and the expression for the signal  $I_c(t)$  (39). We start with the conditioned master equation given in Eq. (28) and apply the unitary transformation as defined in Eq. (29). In the new basis the stochastic master equation can be written in the form

$$d\tilde{\rho}_c = (\mathcal{L}_I + \mathcal{L}_\nu + \tilde{\mathcal{J}})\tilde{\rho}_c dt + \sqrt{\epsilon\Gamma_g} \mathcal{H}[\sigma_{ge} e^{-i\phi}] \tilde{\rho}_c dW(t), \quad (A1)$$

where we divided the total Liouville operator into the three contributions,

$$\mathcal{L}_I(\rho) = -\frac{i}{\hbar} [H_I, \rho] - \frac{\Gamma}{2} (|e\rangle\langle e| \rho + \rho |e\rangle\langle e|),$$

$$\mathcal{L}_\nu(\rho) = -\frac{i}{\hbar} [H_E + H_{\text{int}}, \rho],$$

$$\tilde{\mathcal{J}}(\rho) = \Gamma \tilde{\mathcal{J}}_g(\sigma_{ge} \rho \sigma_{eg}) + \Gamma \tilde{\mathcal{J}}_r(\sigma_{re} \rho \sigma_{er}) \dots$$

The action of the recycling operators,  $\tilde{\mathcal{J}}_{g,r}$ , is defined in Sec. III, Eq. (41).

In the following we write the total density operator in terms of the eigenbasis of  $H_I$  as,  $\tilde{\rho}_c = \sum_{i,j} \tilde{\mu}_{ij} \otimes |i\rangle\langle j|$  with  $i, j \in \{+, -, D\}$ . Reinserting this decomposition into Eq. (A1) we obtain a set of coupled equations for the external operators  $\tilde{\mu}_{ij}$ . By grouping the 9 elements  $\tilde{\mu}_{ij}$  into a single vector  $\tilde{\mu}$ , the resulting set of equations can be written in the form

$$d\tilde{\mu} = (\mathbf{L}_I + \mathbf{L}_\nu + \mathbf{J})\tilde{\mu} dt + \sqrt{\epsilon\Gamma_g} (\mathbf{S} - \text{Tr}\{\vec{V} \cdot \tilde{\mu}\}) \tilde{\mu} dW(t). \quad (A2)$$

The entries of the matrices  $\mathbf{L}_I, \mathbf{L}_\nu, \mathbf{J}, \mathbf{S}$  and the vector,  $\vec{V}$ , can be derived in a straight forward (but lengthy) way by writing

the operators in Eq. (A1) in terms of the states,  $|+\rangle, |-\rangle$  and  $|D\rangle$ . Note that the entries of  $\mathbf{L}_\nu$  and  $\mathbf{J}$  still contain superoperators acting on the external operators  $\tilde{\mu}_{ij}$ .

The goal is to derive an effective equation for the population in the dark state,  $\tilde{\mu}_0 := \tilde{\mu}_{DD}$ , for the parameter regime  $\nu, \Delta_D \ll \Omega_\pm$ . Formally we can combine both conditions and make a series expansion in the trap frequency  $\nu$ . According to the structure of  $\mathbf{L}_I$  we group the external operators  $\tilde{\mu}_{ij}$  into the two vectors  $\tilde{\mu}_1 := (\tilde{\mu}_{D+}, \tilde{\mu}_{+D}, \tilde{\mu}_{D-}, \tilde{\mu}_{-D})^T$  and  $\tilde{\mu}_2 := (\tilde{\mu}_{++}, \tilde{\mu}_{--}, \tilde{\mu}_{+-}, \tilde{\mu}_{-+})^T$ . By ordering the entries of  $\tilde{\mu}$  such that,  $\tilde{\mu} = (\tilde{\mu}_2^T, \tilde{\mu}_1^T, \mu_0)^T$ , the matrices  $\mathbf{L}_I, \mathbf{J}$  and  $\mathbf{L}_\nu$  have the block form

$$\mathbf{L}_I + \mathbf{J} = \begin{pmatrix} L_I^2 + J^2 & 0 & 0 \\ J^{12} & L_I^1 & 0 \\ J^{02} & 0 & 0 \end{pmatrix}, \quad \mathbf{L}_\nu = \begin{pmatrix} L_\nu^2 & L_\nu^{21} & 0 \\ L_\nu^{12} & L_\nu^1 & L_\nu^{10} \\ 0 & L_\nu^{01} & L_\nu^0 \end{pmatrix}.$$

From the structure of  $\mathbf{L}_I$  and  $\mathbf{L}_\nu$  we see that for  $\nu \rightarrow 0$  we have  $\tilde{\mu}_1 \sim \mathcal{O}(\nu)$  and  $\tilde{\mu}_2 \sim \mathcal{O}(\nu^2)$ . Therefore, up to second order in  $\nu$  the equation for  $\tilde{\mu}_0$  is

$$d\tilde{\mu}_0 = (L_\nu^0 \tilde{\mu}_0 + L_\nu^{01} \tilde{\mu}_1 + J^{02} \tilde{\mu}_2) dt + \sqrt{\epsilon \Gamma} \mathbf{g}(\vec{v} \cdot \tilde{\mu}_1 - \text{Tr}\{\vec{v} \cdot \tilde{\mu}_1\} \mu_0) dW(t), \quad (\text{A3})$$

with

$$\vec{v} = \frac{\Omega_L}{\Omega} [e^{i\phi} \cos(\theta), e^{-i\phi} \cos(\theta), e^{i\phi} \sin(\theta), e^{-i\phi} \sin(\theta)]. \quad (\text{A4})$$

The equations for  $\tilde{\mu}_1$  and  $\tilde{\mu}_2$  are given by

$$\dot{\tilde{\mu}}_1 = L_I^1 \tilde{\mu}_1 + L_\nu^{10} \tilde{\mu}_0 + \mathcal{O}(\nu^2),$$

$$\dot{\tilde{\mu}}_2 = (L_I^2 + J^2) \tilde{\mu}_2 + L_\nu^{21} \tilde{\mu}_1 + \mathcal{O}(\nu^3).$$

They can be integrated and up to the relevant orders of  $\nu$  we obtain the formal solution

$$\tilde{\mu}_1 = (L_I^1)^{-1} L_\nu^{10} \tilde{\mu}_0 + \mathcal{O}(\nu^2),$$

$$\tilde{\mu}_2 = (L_I^2 + J^2)^{-1} L_\nu^{21} (L_I^1)^{-1} L_\nu^{10} \tilde{\mu}_0 + \mathcal{O}(\nu^3).$$

Resubstituting these expressions into Eq. (A3) the resulting equation can be written in the form

$$d\tilde{\mu}_0 = -i[\hat{h}_{\text{eff}} \tilde{\mu}_0 - \tilde{\mu}_0 \hat{h}_{\text{eff}}^\dagger] dt + \lambda^2 \Gamma \mathcal{R}(\hat{p} \tilde{\mu}_0 \hat{p}) dt + \sqrt{\frac{\epsilon \Gamma \lambda^2 \Omega_L^2}{\Omega^2}} \mathcal{T}[\hat{p} e^{-i\phi}] \tilde{\mu}_0 dW(t), \quad (\text{A5})$$

with  $\lambda = 2\bar{\eta} \nu g \Omega_L / \Omega^3$ , the non-Hermitian operator

$$\hat{h}_{\text{eff}} = \nu a^\dagger a - \frac{g\lambda}{2} \hat{p} - \lambda^2 \Delta \hat{p}^2 - i \frac{\lambda^2 \Gamma}{2} \hat{p}^2 \quad (\text{A6})$$

and the recycling term,

$$\mathcal{R} = \frac{\frac{\Omega_L^2}{\Omega^2} \tilde{\mathcal{J}}_g + \frac{g^2}{\Omega^2} \tilde{\mathcal{J}}_r}{1 - \frac{g^2}{\Omega^2} \tilde{\mathcal{J}}_g - \frac{\Omega_L^2}{\Omega^2} \tilde{\mathcal{J}}_r}. \quad (\text{A7})$$

In the formal expression for  $\mathcal{R}$  the inversion of operator is justified in the limit of a weak probe field,  $g \ll \Omega_L$  and  $\Gamma_r \ll \Gamma$ . From the stochastic term in Eq. (A5) we see that these conditions also maximize the signal strength for a given decoherence rate  $\lambda^2 \Gamma$ .

In the original basis the evolution for  $\mu_c$  is given by the relation,

$$d\mu_c = \text{Tr}_I\{U d\tilde{\mu}_0 \otimes |D\rangle\langle D| U^\dagger\}. \quad (\text{A8})$$

Due to the overlap  $|\langle r|D\rangle|^2 = g^2/\Omega^2$  the action of  $U$  on the external states reduces to the action of the operator  $\exp(i\bar{\eta}\hat{z}g^2/\Omega^2)$ . Therefore, the only effect of the basis transformation is the cancellation of the term  $\lambda g \hat{p}/2$  in the effective Hamiltonian,  $\hat{h}_{\text{eff}}$ .

For the expression of the measured signal  $I_c(t)$  (39) we can simply repeat the calculations from above. Using the same notation as in Eq. (A2) it can be written as

$$I_c(t) = \epsilon \Gamma_g \text{Tr}\{\vec{V} \cdot \tilde{\mu}\} + \sqrt{\epsilon \Gamma_g} \xi(t). \quad (\text{A9})$$

We see that the first term already appeared in the stochastic master equation (A2) and can be evaluated along the same lines. By multiplying the resulting expression by a factor  $\lambda$ , we obtain the current given in Eq. (39).

## APPENDIX B: ADIABATIC ELIMINATION IN THE LAMB-DICKE LIMIT

In this appendix we derive the conditioned evolution of the external atomic state in the Lamb-Dicke limit,  $\eta_p, \eta_L \ll 1$ . We start with the stochastic Schrödinger equation for the total wave function,  $|\Psi\rangle$ , which includes the state of the atom as well as the state of the electromagnetic environment. For the system Hamiltonian  $H_S$  (52) and the atom-field interaction described in Sec. III it is given by

$$d|\Psi\rangle = \left(-\frac{i}{\hbar} H_S - \frac{\Gamma}{2} |e\rangle\langle e|\right) |\Psi\rangle dt + \sqrt{\Gamma} \int_{-1}^1 du \sqrt{N(u)} e^{-i\eta_p \hat{z} u} \sigma_{ge} dB_u^\dagger(t) |\Psi\rangle. \quad (\text{B1})$$

The noise increment operators  $dB_u^\dagger(t)$  fulfill the Ito rules  $dB_u(t) dB_{u'}^\dagger(t) = \delta(u-u') dt$  and correspond to the emission of photons under an angle  $\alpha = \arccos(u)$  with respect to the  $z$  axis.

We decompose the total wave function in terms of the eigenstates of  $H_I$  as  $|\Psi\rangle = \sum_i |\Psi_i\rangle \otimes |i\rangle$ , with  $i = +, -, D$ . For vanishing Lamb-Dicke parameters,  $\eta_j \rightarrow 0$ , and after some transient deviations the system evolves into the state,  $|\Psi\rangle = |\Psi_D\rangle \otimes |D\rangle$ . The coupling between the external and internal degrees of freedom,  $H_{\text{int}}$ , leads to finite contributions from the bright states which are of the order of the Lamb-Dicke parameter,  $|\Psi_{\pm}\rangle \sim \mathcal{O}(\bar{\eta})$ . In the following we treat the coupling to the excited states in perturbation theory to derive an effective equation for  $|\Psi_D\rangle$  which is valid up to second order in  $\bar{\eta}$ .

In the interaction picture with respect to the external Hamiltonian,  $H_E = \hbar\nu\hat{a}^\dagger\hat{a}$ , the equation for the dark state wave function is

$$d|\Psi_D\rangle \simeq -\frac{\bar{\eta}g}{2}\hat{z}(t)|\Psi_e\rangle dt + \sqrt{\Gamma}\int_{-1}^1 du\sqrt{N(u)}dB_u^\dagger(t)|\Psi_e\rangle, \quad (\text{B2})$$

with  $|\Psi_e\rangle = \cos(\theta)|\Psi_+\rangle + \sin(\theta)|\Psi_-\rangle$ . Note that in this equation we already used the assumption of a weak probe field, and set  $\Omega = \sqrt{\Omega_L^2 + g^2} \approx \Omega_L$ . Under the same assumption the equations for the bright states are given by

$$\frac{d}{dt}\begin{pmatrix} |\Psi_+\rangle \\ |\Psi_-\rangle \end{pmatrix} = -\mathbf{M}\begin{pmatrix} |\Psi_+\rangle \\ |\Psi_-\rangle \end{pmatrix} + \frac{\bar{\eta}g}{2}\hat{z}(t)\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}|\Psi_D\rangle, \quad (\text{B3})$$

where we defined the matrix

$$\mathbf{M} = \begin{pmatrix} i\Omega_+ + \frac{\Gamma}{2}\cos^2(\theta) & \frac{\Gamma}{4}\sin(2\theta) \\ \frac{\Gamma}{4}\sin(2\theta) & i\Omega_- + \frac{\Gamma}{2}\sin^2(\theta) \end{pmatrix}. \quad (\text{B4})$$

Up to first order in  $\bar{\eta}$ , the solution for the excited states is

$$\begin{pmatrix} |\Psi_+\rangle \\ |\Psi_-\rangle \end{pmatrix} = \frac{\bar{\eta}g}{2}\left[\int_{-\infty}^t e^{-\mathbf{M}(t-s)}\hat{z}(s)ds\right]\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}|\Psi_D\rangle. \quad (\text{B5})$$

By inserting the time dependence of the position operator,  $\hat{z}(t) = (\hat{a}e^{-i\nu t} + \hat{a}^\dagger e^{i\nu t})/\sqrt{2}$ , we can evaluate this integral and obtain the evolution of the wave functions,  $|\Psi_{\pm}\rangle$ . This solution is then reinserted into Eq. (B2) to get the equation of the dark state wave function up to order  $\bar{\eta}^2$ . As long as the final dynamics in the interaction picture is slow compared to the trap frequency,  $\nu$ , we can use the rotating wave approximation and neglect terms proportional to  $e^{\pm i2\nu t}$ . The resulting equation is then given by

$$d|\Psi_D\rangle = -\frac{\bar{\eta}^2 g^2}{8}[\tilde{I}(\nu)\hat{a}\hat{a}^\dagger + \tilde{I}(-\nu)\hat{a}^\dagger\hat{a}]|\Psi_D\rangle dt + \sqrt{\frac{\bar{\eta}^2 g^2}{8}}\Gamma\int_{-1}^1 du\sqrt{N(u)}\tilde{C}(t)dB_u^\dagger(t)|\Psi_D\rangle, \quad (\text{B6})$$

where we set  $\tilde{C}(t) = [\tilde{I}(-\nu)\hat{a}(t) + \tilde{I}(\nu)\hat{a}^\dagger(t)]$ , and defined the function  $\tilde{I}(\nu)$  by

$$\tilde{I}(\nu) := [\cos(\theta), \sin(\theta)]\left[\int_0^\infty e^{-\mathbf{M}\tau}e^{-i\nu\tau}d\tau\right]\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}. \quad (\text{B7})$$

Apart from the motional state of the atom, the wave function  $|\Psi_D\rangle$  still includes the full state of the electromagnetic environment. To obtain the conditioned dynamics for the external density operator,  $\mu_c$ , we first decompose the set of noise increment operators into the two contributions,

$$dB_p^\dagger(t) = \frac{1}{\sqrt{\epsilon}}\int_{1-\epsilon}^1 du\sqrt{N(u)}dB_u^\dagger(t),$$

$$dB_\perp^\dagger(t) = \frac{1}{\sqrt{1-\epsilon}}\int_{-1}^{1-\epsilon} du\sqrt{N(u)}dB_u^\dagger(t).$$

As in Sec. III the parameter  $\epsilon$  determines the fraction of the photons which are scattered into the mode of the probe beam. The increment operator  $dB_p^\dagger(t)$  obeys the Ito rule,  $dB_p(t)dB_p^\dagger(t) = dt$ , and corresponds to the emission of photons, which are focused on the detector.

A rigorous, but rather technical way to convert the stochastic Schrödinger equation into a stochastic master equation can be found in Refs. [26,30]. A convenient shortcut is, to first derive the unconditioned master equation for  $\mu = \text{Tr}_{EM}\{|\Psi_D\rangle\langle\Psi_D|\}$  and then perform a partial 'unravelling' of this master equation as discussed in Ref. [28]. For a phase  $\phi$  of the local oscillator we finally get the equation

$$d\mu_c = -i\frac{\bar{\eta}^2 g^2}{8}\text{Im}[\tilde{I}(-\nu) + \tilde{I}(\nu)][\hat{a}^\dagger\hat{a}, \mu_c]dt + \frac{\bar{\eta}^2 g^2}{4}(\text{Re}[\tilde{I}(-\nu)]\mathcal{D}[\hat{a}] + \text{Re}[\tilde{I}(\nu)]\mathcal{D}[\hat{a}^\dagger])\mu_c dt + \sqrt{\epsilon\Gamma\frac{\bar{\eta}^2 g^2}{8}}\mathcal{H}[\tilde{C}(t)e^{-i\phi}]\mu_c dW(t), \quad (\text{B8})$$

The measured current has the form

$$I_c(t) = \epsilon\Gamma\frac{\bar{\eta}^2 g^2}{8}\langle\tilde{C}(t)e^{-i\phi} + \tilde{C}^\dagger(t)e^{i\phi}\rangle_c + \sqrt{\epsilon\Gamma\frac{\bar{\eta}^2 g^2}{8}}\xi(t). \quad (\text{B9})$$

To make a comparison to the results of Sec. III and Sec. IV we introduce the rescaled function  $I(\nu)$  by setting  $I(\nu) = \tilde{I}(\nu)\Omega^4/(8\nu^2\Gamma)$ . With this definition and a rescaling of the current we finally obtain Eqs. (54) and (55).

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