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Quantum feedback cooling of a single trapped ion in front of a mirror

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We develop a theory of quantum feedback cooling of a single ion trapped in front of a mirror. By monitoring the motional sidebands of the light emitted into the mirror mode we infer the position of the ion, and act back with an appropriate force to cool the ion. We derive a feedback master equation along the lines of the quantum feedback theory developed by Wiseman and Milburn, which provides us with cooling times and final temperatures as a function of feedback gain and various system parameters.

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I. INTRODUCTION

Laser cooling and trapping of single ions [1–3] is one of the highlights in the development of quantum optics during the last two decades. Single trapped ions are a laboratory paradigm of a quantum system, which can be prepared and controlled on the single quantum level, and whose time evolution can be monitored continuously by observing the scattered light in photodetection or homodyne measurements [1]. By continuous observation of a single quantum system [4,5] we learn the state of the system, as described by a conditional system density matrix $\rho_c(t)$, and based on this knowledge we can act back on the system, giving rise to quantum feedback control of the system of interest [6–10]. In the present paper we present a theory of *quantum feedback cooling of a single trapped ion*: by extracting from the scattered light the position of the ion in the trap, we implement a feedback loop on the system in the form of a damping force with the purpose of cooling the ion motion in the trap. Development of this theory is not only of fundamental interest in quantum optics, but the particular setup studied is motivated by ongoing experimental efforts [11,12] to implement quantum feedback cooling of single trapped ions in laboratory. Indeed the present theoretical results provides a quantitative basis for the understanding of these experiments [12].

The particular setup studied in the present paper is a single laser cooled trapped ion in front of a mirror [13], as illustrated in Fig. 1, and motivated by present experiments [11,12]. A single ion is stored a distance L from a mirror in a harmonic trapping potential. The ion is assumed to be a two-level system weakly driven near resonance by laser light. Light is scattered into both the mirror mode, as well as the other other “background” modes of the radiation field. By detuning the laser on the red side of the atomic transition, the ion is laser cooled to a temperature corresponding to Doppler limit, where the mean occupation of the trap levels is much larger than one (i.e., far away from the sideband cooling limit to the ground state of the trap). Motion of the ion adds sidebands of the light scattered into the mirror mode displaced by the trap frequency. Observing the scattered light of

these motional sidebands allows us to infer the position of the ion in the sense of continuous measurement theory, and feed back a damping force proportional to the momentum to implement quantum feedback cooling. In this paper we will first formulate a continuous measurement theory to read the position of the trapped ion from the scattered light using the language of stochastic Schrödinger equations [4,5]. Building on general quantum feedback theory formulated by Wiseman and Milburn [6,7], we will then derive a quantum feedback master equation for the motion of the trapped ion. This will allow us to study the dynamics and limits of quantum feedback cooling.

For the setup studied in this paper the continuous readout of the ion position is based on light scattering into the mirror mode, with additional photons scattered into all other “background modes” of the radiation field. Spontaneous emission is intrinsically associated with a momentum recoil of the ion, which perturbs the ion motion, i.e., contributes a heating mechanism for the ion. In a parallel paper [14] we study a quantum feedback scheme based on a *dispersive readout of the velocity* of the trapped ion to avoid this unwanted heating. It is based on the large variation of the index of refraction with the Doppler effect near a *dark state resonance in an atomic Λ system* (based on electromagnetically induced transparency).

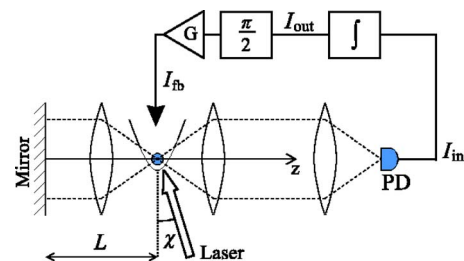


FIG. 1. (Color online) Physical setup: L is the distance between the ion trap center and the mirror, χ is the incident laser angle. The light is collected in the photodetector PD. The feedback circuit consists of a bandpass filter, a phase shifter, and an amplifier. The current I_{fb} is fed back to electrodes creating an additional potential for the ion. The mirror axis is equal to the z axis.

The paper is structured as follows. Section II presents the basic dynamic equations for the motion of an ion in front of a mirror. Quantum feedback equations are formulated in Sec. III, while results are presented in Sec. IV.

II. MODEL AND BASIC EQUATIONS

In this section we will develop the basic equations for continuous measurement of the photons in the mirror mode of the electromagnetic field. We will start with a detailed description of our model in terms of a Schrödinger equation for the coupled atom-bath system and the exciting laser. Continuous measurement theory provides us with a quantum stochastic Schrödinger equation and hence a quantum stochastic master equation in the Lamb-Dicke limit, where we adiabatically eliminate the excited state from the two-level atom. We will then derive the photocurrent obtained by detecting mirror mode photons and the corresponding stochastic master equation for the conditional density operator in the white noise (diffusive) limit.

A. Single trapped ion in front of a mirror

We consider a single trapped ion which is placed at a distance L from a mirror as indicated in Fig. 1 [1,15–17]. For the harmonic motion we assume a 1D model in the z direction (identical to the mirror axis). The harmonic trap has an oscillation frequency ν_T , and we denote the destruction (creation) trap operator by $a(a^\dagger)$. The electronic degrees of the ion form a two-level atom with atomic transition frequency ω_{eg} , with ground state $|g\rangle$ and excited state $|e\rangle$. We drive the two-level system with a laser with frequency ω_L which couples the ground to the excited state with the Rabi frequency Ω and a detuning from the atomic resonance $\Delta_L = \omega_L - \omega_{eg}$. The atomic system Hamiltonian can thus be written as

$$H_{\text{sys}} = \nu_T a^\dagger a - \Delta_L |e\rangle\langle e| - \frac{1}{2} \Omega (e^{ik_{\text{eff}}\hat{z}} |e\rangle\langle g| + \text{H.c.}). \quad (2.1)$$

Note that in this paper we set $\hbar = 1$. In the interaction term we allow for a laser field incident at an angle χ with respect to an axis normal to the z axis. The momentum recoil due to absorption of a laser photon is represented by $k_{\text{eff}}\hat{z} = \eta \sin \chi (a + a^\dagger) \equiv \tilde{\eta}(a + a^\dagger)$ where the Lamb-Dicke parameter $\eta = 2\pi a_0/\lambda$ is the ratio of the size of the ground state and the laser wavelength. Due to the geometry of the system in consideration, the (quantized) electric field consists of two contributions, $\vec{E}^{(+)} = \vec{E}_m^{(+)} + \vec{E}_b^{(+)}$, where the $\vec{E}_m^{(+)}$ denotes the modes restricted by the boundary condition of the mirror and $\vec{E}_b^{(+)}$ the remaining background modes [16,17]. We adopt a 1D model for the mirror mode and write for the electric field operator

$$\vec{E}_m^{(+)}(z) = i \int_0^\infty d\omega \alpha_\omega \vec{e} \sin[k(\omega)z] b_m(\omega), \quad (2.2)$$

with α_ω a normalization factor for the mode function. The internal states of the atom couple to the vacuum field by an electric dipole transition. Denoting by \vec{d} the dipole matrix element, and introducing Pauli operator notation for the two

level system, $\sigma_- = |g\rangle\langle e|$, the system-bath coupling Hamiltonian is

$$H_{\text{int}} = -\vec{d}(\vec{E}_b^{(+)}(\hat{z}) + \vec{E}_m^{(+)}(\hat{z}))\sigma_- + \text{H.c.} \quad (2.3)$$

The total Hamiltonian for the ion coupled to the radiation field is

$$H = H_{\text{sys}} + H_{\text{bath}} + H_{\text{int}}. \quad (2.4)$$

Here H_{bath} is the free Hamiltonian for the radiation field. We write this Hamiltonian as the sum of a Hamiltonian for the mirror and the background modes $H_{\text{bath}} = H_m + H_b$. In our 1D model the mirror mode Hamiltonian has the form $H_m = \int d\omega \omega b_m^\dagger(\omega) b_m(\omega)$ with $b_m(\omega)$ photon destruction operators, satisfying bosonic commutation relations $[b_m(\omega), b_m^\dagger(\omega')] = \delta(\omega - \omega')$. Similar expression can be given for the background modes.

In analyzing this problem we are interested in the situation where the time delay $\tau_M = 2L/c$ of the emitted light bouncing from the mirror back to the atom is much shorter than the system time scales, in particular the spontaneous emission time from the excited state, $\tau_M \ll 1/\Gamma$, and the timescales associated with the laser interactions $\tau \ll 1/\Omega$, $1/|\Delta_L|$. This justifies the Markov approximation for the emission into the mirror modes, where we refer to [16] for a complete analysis.

In the following we will denote the total spontaneous emission rate of the atom by $\Gamma = \Gamma_m + \Gamma_b$. Here $\Gamma_m = \varepsilon\Gamma$ with ε the fraction of the solid angle covered by the lens is the emission rate into mirror mode, and $\Gamma_b = (1 - \varepsilon)\Gamma$ the emission rate into the background modes.

B. Quantum stochastic Schrödinger equation

The dynamics of our model is summarized in the Schrödinger equation

$$\begin{aligned} |\dot{\Psi}(t)\rangle = & \left\{ -iH_{\text{sys}} + \sqrt{\Gamma_m} \sigma_- \sin[k_{eg}(L + \hat{z})] b_m^\dagger(t) + \text{H.c.} \right. \\ & \left. + \sqrt{\Gamma_b} \int_{-1}^{+1} du \sqrt{N(u)} \sigma_- e^{-iuk_{eg}\hat{z}} b_u^\dagger(t) + \text{H.c.} \right\} |\Psi(t)\rangle. \end{aligned} \quad (2.5)$$

We choose to formulate the problem in the language of a *quantum stochastic Schrödinger equation* (QSSE) [5], which allows for a direct connection with continuous measurement of the scattered light, and provides a direct link to quantum feedback theory developed in the following subsections.

In Eq. (2.5) $|\Psi(t)\rangle$ is the Schrödinger state vector of the combined atom-field system, i.e., the laser-driven trapped ion including the mirror and background modes of the radiation field. The first term on the right-hand side (RHS) is the time evolution due to the system Hamiltonian (2.1).

The second and third line describe the interaction of the two-level atom with the mirror mode and the background modes, respectively. We assume that these radiation modes are initially in the vacuum state. In writing Eq. (2.5) we have transformed to an interaction picture with respect to the free

Hamiltonian of the radiation field H_B [5]. As a result, we have introduced bath operators for the mirror mode $b_m(t) = 1/\sqrt{2\pi} \int d\omega b_m(\omega) e^{-i\omega t}$. In the Markovian limit these operators satisfy bosonic commutation relations

$$[b_m(t), b_m^\dagger(s)] = \delta(t-s). \quad (2.6)$$

In a quantum Langevin formulation [5] $b_m(t)$ represents a quantum noise operator. Thus the second line of (2.5) describes the emission of photons by the atom into the mirror mode, with the center of the ion trap displaced a distance L from the mirror. We note that the motion of the ion couples to the light via the recoil, as seen by the appearance of \hat{z} in the mirror mode function. This coupling imparts information of the ion motion on the light emitted in the mirror mode. In the following subsections we will analyze this scattered light to continuously monitor the atomic motion, with the goal of implementing a feedback loop to cool the ion. The coupling strength to the mirror mode is proportional to the square root of the spontaneous emission probability into the mirror mode $\Gamma_m \equiv \varepsilon \Gamma$ with ε the fraction of the solid angle (typically ε is much smaller than one).

The third line in Eq. (2.5) represents spontaneous emission of the ion into the background modes. This is a coupling term familiar from the theory of laser cooling of two-level atoms [15]. Spontaneous emission into the background mode is again associated with a recoil of the ion motion. In our 1D model for the motion of the trapped ion, it is the projection of this momentum on the z axis which is the relevant momentum transfer. Denoting by θ the angle between the emitted photon and the z axis, and $u = \cos \theta$, we associate the transition for the excited state to the ground state including the momentum transfer with the operator $e^{iuk_{eg}\hat{z}}\sigma_-$, where $k_{eg} \equiv \omega_{eg}/c \approx k_L$. Spontaneous photons can be emitted in all directions into the background modes consistent with the dipole radiation pattern of the given electronic transition. We denote this (normalized) angular dependence by $N(u)$. Thus the integral over u in the last line of Eq. (2.5) realizes photon emission into all of these possible directions. The operators $b_u(t)$ are again photon destruction (or noise) operators associated with these emission directions. They satisfy commutation relations

$$[b_u(t), b_{u'}^\dagger(s)] = \delta(u-u')\delta(t-s), \quad (2.7)$$

and commute with the mirror bath operators $b_m(t)$ introduced above. The coupling strength to the background modes is proportional to $\sqrt{\Gamma_b} \equiv \sqrt{(1-\varepsilon)\Gamma}$. For red laser detuning $\Delta_L < 0$ the cycle of laser excitation followed by spontaneous emission into the background mode leads to laser cooling.

C. Ito form of the quantum stochastic Schrödinger equation

To give a meaning to the white noise limit [compare Eqs. (2.6) and (2.7)], we must interpret the Schrödinger equation

(2.5) as a quantum stochastic *Stratonovich* equation [5]. As usual, it is more convenient to work with an Ito form, where Wiener noise increments

$$dB_{m,u}(t) = \int_t^{t+dt} ds b_{m,u}(s) \quad (2.8)$$

“point to the future,” i.e., are statistically independent of $|\Psi(t)\rangle$. These Wiener noise increments satisfy the Ito table

$$dB_m(t)dB_m^\dagger(t) = dt, \quad (2.9a)$$

$$dB_u(t)dB_{u'}^\dagger(t) = \delta(u-u')dt, \quad (2.9b)$$

which follows from Eqs. (2.6) and (2.7), the other entries of the Ito table being zero. The resulting Ito QSSE is

$$(I): \quad d|\Psi(t)\rangle = \left[-iH_{\text{eff}}dt + \sqrt{\Gamma_m}C_m(\hat{z})dB_m^\dagger(t) + \sqrt{\Gamma_b} \int_{-1}^{+1} du \sqrt{N(u)}C_u(\hat{z})dB_u^\dagger(t) \right] |\Psi(t)\rangle. \quad (2.10)$$

Here, we have introduced the “jump operators”

$$C_u(\hat{z}) = e^{-iuk_{eg}\hat{z}}\sigma_-, \quad (2.11a)$$

$$C_m(\hat{z}) = \sin(k_{eg}(L+\hat{z}))\sigma_-, \quad (2.11b)$$

which are associated with the emission of a photon in the background modes and the mirror modes, respectively. Furthermore, we have defined an *effective non-Hermitian system Hamiltonian*

$$H_{\text{eff}} = H_{\text{sys}} - \frac{i}{2} \{ \Gamma_b + \Gamma_m \sin^2[k_{eg}(L+\hat{z})] \} |e\rangle\langle e|. \quad (2.12)$$

The non-Hermitian part of H_{eff} arises from the Ito correction in the conversion process. Physically, it corresponds to the radiation damping of the excited state due to the total radiation field. We also note that the photon absorption terms have disappeared in Eq. (2.10) due to $dB_{m,u}(t)|\Psi(t)\rangle = 0$. This follows from our assumption of an initial vacuum state.

D. Quantum stochastic master equation

We are interested in the time evolution of our system where the photons emitted in the mirror mode are detected by a photon counter, while the background modes remain unobserved. Therefore, we are only interested in the dynamics of the reduced density operator $\hat{W}(t) \equiv \text{Tr}_b\{|\Psi(t)\rangle\langle\Psi(t)|\}$ where we trace over the background modes of the radiation field. We emphasize that $\hat{W}(t)$ still contains all the degrees of freedom of the mirror modes, in addition to the internal and external atomic dynamics.

Using Ito calculus we obtain the quantum stochastic master equation (QSME)

$$\begin{aligned}
\text{(I): } d\hat{W}(t) = & -i[H_{\text{eff}}\hat{W}(t) - \hat{W}(t)H_{\text{eff}}^\dagger]dt \\
& + \Gamma_m \mathcal{J}[C_m(\hat{z})]dB_m^\dagger(t)\hat{W}(t)dB_m(t) \\
& + \sqrt{\Gamma_m}[C_m(\hat{z})dB_m^\dagger\hat{W}(t) + \hat{W}(t)C_m^\dagger(\hat{z})dB_m] \\
& + \Gamma_b \int_{-1}^{+1} du N(u) \mathcal{J}[C_u(\hat{z})]\hat{W}(t), dt \quad (2.13)
\end{aligned}$$

with H_{eff} defined in Eq. (2.12). For the ‘‘recycling terms’’ we use the notation

$$\mathcal{J}[c]\rho \equiv c\rho c^\dagger. \quad (2.14)$$

Before proceeding we note that for $\varepsilon=0$, i.e., no coupling to the mirror modes, Eq. (2.13) reduces to the standard master equation for 1D laser cooling of a two-level atom [5]. In this case \hat{W} is only an atomic density operator containing the internal and motional dynamics. For $\varepsilon \neq 0$, we still have a stochastic equation with the mirror bath degrees of freedom included.

E. Adiabatic elimination of the excited state and Lamb-Dicke limit

We will simplify the above QSSE (2.10) and QSME (2.13) with two assumptions. First, we assume weak laser excitation to the excited state, $\Omega \ll \max(\Gamma, |\Delta|)$. Second, we assume a small Lamb-Dicke parameter $\eta \equiv 2\pi a_0/\lambda \ll 1$ (tight trap): this allows us to expand the exponents $e^{ik\hat{z}} \equiv e^{i\eta(a+a^\dagger)} = 1 + i\eta(a+a^\dagger) + O(\eta^2)$. Both of these assumptions are well satisfied in present experiments [1]

To eliminate the weakly populated excited level, we go back to Eq. (2.10) and expand the state vector $|\Psi(t)\rangle$ into ground state and excited state components,

$$|\Psi(t)\rangle \equiv |\psi_g(t)\rangle \otimes |g\rangle + |\psi_e(t)\rangle \otimes |e\rangle. \quad (2.15)$$

As shown in Appendix A we can eliminate $|\psi_e(t)\rangle$ in perturbation theory in the Ito QSSE (2.10) to obtain an effective equation for $|\psi_g(t)\rangle$. In a similar way as for Eq. (2.13) we obtain a QSME for the partially reduced density operator

$$\hat{w}(t) \equiv \text{Tr}_b\{|\psi_g(t)\rangle\langle\psi_g(t)|\}, \quad (2.16)$$

given by

$$\begin{aligned}
\text{(I): } d\hat{w}(t) = & -i[h_{\text{eff}}\hat{w}(t) - \hat{w}(t)h_{\text{eff}}^\dagger]dt \\
& + \gamma \mathcal{J}[c_m(\hat{z})]dB_m^\dagger(t)\hat{w}(t)dB_m(t) \\
& + \sqrt{\gamma}[c_m(\hat{z})dB_m^\dagger\hat{w}(t) + \hat{w}(t)c_m^\dagger(\hat{z})dB_m(t)] \\
& + \mathcal{L}_b\hat{w}(t)dt. \quad (2.17)
\end{aligned}$$

The first three lines give the dynamics of the ion motion coupled to the mirror mode. The fourth line describes the traced-out action of the background mode on the ion motion, i.e., laser cooling of the ion.

In Eq. (2.17) we have defined an effective Hamiltonian acting only on the motional states of the ion,

$$h_{\text{eff}} = H_T - \frac{i}{2}\gamma c_m^\dagger(\hat{z})c_m(\hat{z}), \quad (2.18)$$

where we expand the eliminated jump operators to second order in the Lamb-Dicke limit with the center of the trap at $k_{\text{eg}}L = \pi/4$:

$$c_m(\hat{z}) \approx \frac{1}{\sqrt{2}} \left(1 + \eta(a+a^\dagger) - \frac{1}{2}\eta^2(a+a^\dagger)^2 \right). \quad (2.19)$$

The parameter

$$\gamma = \varepsilon \Gamma \frac{\Omega^2}{4} \frac{1}{\Delta_L^2 + \frac{\Gamma}{2}} \quad (2.20)$$

is the optical pumping rate into the mirror mode. The first three lines of Eq. (2.17) thus describe the motional state coupled via laser excitation followed by spontaneous emission to the mirror mode.

The Liouvillian \mathcal{L}_b in the fourth line of Eq. (2.17) is the standard laser cooling Liouvillian for weak field excitation and in the Lamb-Dicke limit [1,2,15],

$$\begin{aligned}
\mathcal{L}_b\hat{w}(t) = & A_- \mathcal{D}[a]\hat{w}(t) + A_+ \mathcal{D}[a^\dagger]\hat{w}(t) \\
\equiv & \Gamma_{\text{eff}}(N+1)\mathcal{D}[a]\hat{w}(t) + \Gamma_{\text{eff}}N\mathcal{D}[a^\dagger]\hat{w}(t), \quad (2.21)
\end{aligned}$$

where we have used the notation

$$\mathcal{D}[c]\rho \equiv c\rho c^\dagger - \frac{1}{2}(c^\dagger c\rho + \rho c^\dagger c). \quad (2.22)$$

The rates

$$A_\pm = \eta^2 \frac{\Omega^2}{4} \Gamma_b \left(\frac{\sin^2 \chi}{(\Delta_L \mp \nu_T)^2 + \frac{\Gamma^2}{4}} + \frac{\alpha}{\Delta_L^2 + \frac{\Gamma^2}{4}} \right) \quad (2.23)$$

have the meaning of cooling (heating) terms for red laser detuning $\Delta_L < 0$. With $\Gamma_{\text{eff}} = A_- - A_+ > 0$ and for $\Delta_L < 0$ we have

$$N = \frac{A_+}{A_- - A_+}, \quad (2.24)$$

which is the final mean trap occupation established by laser cooling (alone). We have also used the abbreviation $\alpha = \int du u^2 N(u)$ for the dipole transition parameter and χ is the incident angle of the laser beam. With these definitions the mirror mode optical pumping rate (2.20) can be written as $\gamma = \varepsilon N \Gamma_{\text{eff}} / (1 + \alpha) \eta^2$, and from $\Gamma_{\text{eff}} \propto \sin^2 \chi$ and $N \propto 1/\sin^2 \chi$ we see that this pumping rate is independent from the angle of the incoming laser beam.

In the following we will study a scenario [12] where the laser cooling establishes a steady state with a mean trap occupation $N \gg 1$ (i.e., far from the ground state), as represented by the second line in Eq. (2.17). This is the limit of Doppler cooling, which is obtained if $\Gamma \gg \nu_T$. The minimally obtainable steady state energy in this limit is $\hbar\Gamma(\alpha+1)/2$. By observing the spontaneous emission into the mirror mode

[see first two lines of Eq. (2.17)], we will infer the position of the atom to apply a feedback loop to cool the system (far) below the laser cooling limit.

For completeness we note that in the case where the mirror mode is not observed, the reduced system density operator $\rho(t) \equiv \text{Tr}_m\{\hat{w}(t)\}$ obeys the master equation

$$\begin{aligned} \dot{\rho}(t) &= -i[H_T, \rho(t)] + \gamma \mathcal{D}[c_m(\hat{z})]\rho(t) + \mathcal{L}_b \rho(t) \\ &\equiv -i[H_T, \rho(t)] + \mathcal{L}_{LC} \rho(t) \equiv \mathcal{L}_0 \rho(t), \end{aligned} \quad (2.25)$$

which contains the dynamics from the free ion motion, and the dissipative dynamics from the emission into the mirror mode and laser cooling. In a second order expansion in terms of η , we have

$$\mathcal{D}[c_m(\hat{z})] = \eta^2 \cos^2(k_{eg}L) \mathcal{D}[a + a^\dagger] + O(\eta^3), \quad (2.26)$$

which, multiplied by γ , is typically much smaller than $\Gamma_{\text{eff}}N$ and thus the corrections in the heating and cooling rates will be neglected here.

F. Continuous observation of the mirror mode

We measure the photons emitted into the mirror modes by a photon counter as shown in Fig. 1. We denote by $N_c(t)$ the number of photon counts at time t . A particular count trajectory is characterized the photon detection times t_1, t_2, \dots . Our knowledge of the state of the system, given by the internal and external degrees of the ion, for a given count trajectory is represented by a conditional density matrix $\rho_c(t)$ [5].

Given the state of the system at time t , $\rho_c(t)$, the detection of a mirror mode photon in a time interval $(t, t+dt)$ is associated with a quantum jump of the atom described by

$$\rho_{c,\text{jump}}(t+dt) = \frac{\mathcal{J}[c_m]\rho_c(t)}{\text{Tr}\{\mathcal{J}[c_m]\rho_c(t)\}}, \quad (2.27)$$

where according to (2.11b) the atom returns to from the excited state to the ground state, and momentum is transferred to the ion motion in accordance with the mirror mode function. In the case of no observed photon, the system evolves with the effective non trace-preserving Liouvillian L_0

$$\rho_{c,\text{no jump}}(t+dt) = (1 + L_0 dt)\rho_c(t), \quad (2.28)$$

where

$$L_0 \rho \equiv -i[h_{\text{eff}}\rho - \rho h_{\text{eff}}^\dagger] + \mathcal{L}_b \rho,$$

and h_{eff} is defined in Eq. (2.18). The expected number of counts in the interval $(t, t+dt]$ is with $dN_c(t) = N_c(t+dt) - N_c(t)$

$$\langle dN_c(t) \rangle = p_{\text{emission}}^{(t,t+dt]} = \gamma \text{Tr}_{\text{sys}}\{\mathcal{J}[c_m]\rho_c(t)\} dt \quad (2.29)$$

In view of $dN_c(t) = 0$ or 1, for this point process we have the Ito table $dN_c^2(t) = dN_c(t)$ and $dN_c(t)dt = 0$.

We can summarize the above *a posteriori* time evolution in an Ito stochastic Schrödinger equation (see, e.g. [5])

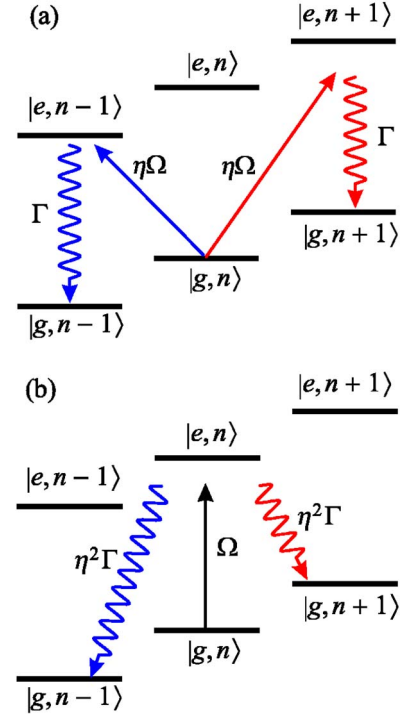


FIG. 2. (Color online) Contributing processes in the master equation at low intensity. (a) shows the cooling and heating terms due to the coupling of the laser to the motion and (b) shows the diffusion terms due to spontaneous emission [18].

$$\begin{aligned} \text{(I): } d\rho_c(t) &= \mathcal{L}_0 \rho_c(t) dt + \left(\frac{\mathcal{J}[c_m]\rho_c(t)}{\text{Tr}_{\text{sys}}\{\mathcal{J}[c_m]\rho_c(t)\}} - \rho_c(t) \right) \\ &\quad \times (dN_c(t) - \gamma \text{Tr}_{\text{sys}}\{\mathcal{J}[c_m]\rho_c(t)\} dt), \end{aligned} \quad (2.30)$$

where \mathcal{L}_0 is defined in (2.25). This equation gives the time evolution of the conditional density matrix of the ion $\rho_c(t)$ for a particular count trajectory. Not observing, i.e., tracing over the mirror mode, is equivalent to taking the ensemble average over all count trajectories in (2.30). In this case, we recover the master equation $\dot{\rho}(t) = \mathcal{L}_0 \rho(t)$ for the *a priori* dynamics [5].

G. Diffusion approximation

In the previous subsection we considered photon counting of the light emitted in the mirror modes, and the associated time evolution of the system described by the conditional density operator $\rho_c(t)$. We are interested in learning the motion (position) of the atom from the scattered light in the sense of continuous measurement. The goal is to use this information to control the motion of the atom, and eventually act back on the atom to cool it.

The scattered light of a weakly driven trapped atom [18] consists of (i) a strong elastic component at the frequency of the driving laser (see vertical transitions in Fig. 2), and (ii) weak motional sidebands at the trap frequency ν_T suppressed by the Lamb-Dicke parameter η . The information on the motion of the atom is encoded in the ‘‘motional sidebands.’’ We find it convenient to formulate the problem in a way,

where we focus directly on the contributions of these sidebands to the photon count signal. The physical picture is that the elastic component acts like a “(strong) local oscillator” which beats with the “(weak) light emitted from the sidebands.” This situation is reminiscent of homodyne measurements in quantum optics [5,4], and will lead in the following to a description in terms of a *diffusive stochastic process* rather than a point process associated with the photon counting described above. The formal expansion parameter is $\eta \ll 1$ (Lamb-Dicke limit).

From the previous subsection we know that the mean number of photon counts in $(t, t+dt]$ is

$$\langle dN_c(t) \rangle = \frac{kL=\pi/4}{2} \gamma dt + \gamma \eta \langle a + a^\dagger \rangle_c(t) dt + O(\eta^2). \quad (2.31)$$

The first term is elastic scattering. The second term, which is first order in η , is proportional to $\tilde{z} \equiv a + a^\dagger$, i.e., includes information on the ion motion. Here and in the following we take the center of the trap to be on the slope of the standing wave, i.e., $k_{eg}L = \pi/4$.

Following the analysis of homodyne detection [4,5], we split the stochastic variable $dN_c(t)$ into a deterministic and a remaining stochastic part, thus defining $dY_c(t)$,

$$dN_c(t) \equiv \frac{1}{2} \gamma dt + \eta dY_c(t), \quad (2.32)$$

and we can show (cf. Appendix B) that $dY_c(t)$ is a Gaussian stochastic variable with non-zero mean, i.e.,

$$dY_c(t) = \sqrt{\gamma/2} dW(t) + \gamma \langle \tilde{z} \rangle_c(t) dt,$$

with $dW(t)$ a Wiener increment satisfying $dW^2(t) = dt$.

This leads us to define a photocurrent where we subtract the large constant contribution from the elastic scattering process,

$$I_c(t) \equiv \eta \frac{dY_c(t)}{dt} = \gamma \eta \langle \tilde{z} \rangle_c(t) + \sqrt{\frac{\gamma}{2}} \xi(t), \quad (2.33)$$

with $\xi(t)$ Gaussian white noise $\langle \xi(t) \xi(t') \rangle = \delta(t-t')$ (shot noise). We see that $I_c(t)$ follows $\langle \tilde{z} \rangle_c(t)$ and thus represents a continuous measurement of the position of the ion. The information on the motion is contained in the sidebands of the current, i.e., in the frequency components centered around $\pm \nu_T$.

In the diffusive approximation the conditional density matrix $\rho_c(t)$ [4,5] obeys

$$(I): \quad d\rho_c(t) = \left[\mathcal{L}_0 dt + \sqrt{\frac{\gamma}{2}} dW(t) \mathcal{H}_m \right] \rho_c(t), \quad (2.34)$$

where

$$\mathcal{H}_m \rho_c(t) = 2\eta [\tilde{z} \rho_c(t) + \rho_c(t) \tilde{z} - 2\langle \tilde{z} \rangle_c(t) \rho_c(t)] \quad (2.35)$$

and Eq. (2.34) is derived from (2.30) in the diffusive limit $\eta \ll 1$ (cf. Appendix B).

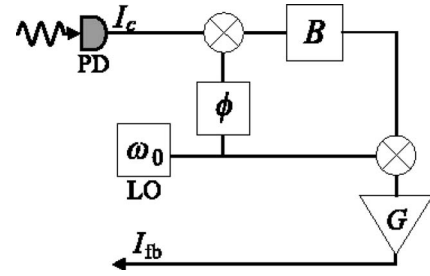


FIG. 3. Electronic setup of the feedback loop as modeled in this paper. PD is the photodetector, LO is the local oscillator, the B box is the band pass (BP) filter, and the ϕ box is the phase shift. The LO signal is mixed to the current and subtracted after the BP filter.

III. QUANTUM FEEDBACK COOLING

In the previous section we have reformulated the continuous observation of the ion motion through spontaneous light scattering into mirror modes in a form reminiscent of homodyne detection. This will allow us below to study feedback cooling of trapped ions building on the *Wiseman-Milburn theory of quantum feedback* [6,7].

In Eq. (2.33) we have obtained a current which is proportional to the mean value of the *position* of the atom. We want to use this information to feed back an appropriate force proportional to the *momentum* to damp the motional state of the atom [11,12]. The information about the position is encoded in the motional sidebands of the current. In a harmonic trap of known frequency any combination of the average position and momentum can be obtained by shifting the sideband current by a phase of ϕ , if the trap frequency is much faster than any other (cooling) timescale in the problem (weak coupling limit). This phase ϕ can be controlled electronically, and for $\phi = \pi/2$, the shifted current follows the momentum. A force, which is proportional to this current, can damp the motion of the ion.

A. Feedback current

We model the feedback circuit as shown in Fig. 3. First, the signal $I_c(t)$ given by Eq. (2.33) is mixed with a local oscillator of frequency $\omega_0 \approx \nu_T$ to shift the signal of the motional sideband to zero frequency. Then the current is sent through a band pass filter of width B to cut off rapidly oscillating terms. The filter is described by a filter function $Z(\omega)$, centered around zero frequency. At the end the signal is mixed again with the local oscillator and amplified by a factor G . The feedback current can then be written as

$$I_{fb,c}(t) = G \cos(\omega_0 t) \int_{-\infty}^t d\tau \tilde{Z}(t-\tau) \cos(\omega_0 \tau + \phi) I_c(\tau), \quad (3.1)$$

where $\tilde{Z}(\tau)$ is the Fourier transform of the band pass function $Z(\omega)$. The feedback Hamiltonian is specified in the next subsection.

To evaluate the expression for the current, it is convenient to change to a basis which is rotating with the frequency of the local oscillator ω_0 by applying the unitary transformation

$U \equiv \exp(-i\omega_0 a^\dagger at)$. The evolution timescale of the density operator in this new frame, $\tilde{\rho}_c(t) \equiv U\rho_c(t)U^\dagger$ is determined by the detuning $\delta = \omega_0 - \nu_T$ and the cooling rates $G\gamma, \Gamma_{\text{eff}}$. Under the assumption, that these frequencies are smaller than the filter bandwidth B , the feedback current is given by

$$I_{\text{fb},c}(t) = G \left[\gamma \eta \langle X_\phi \rangle_c^I(t) + \sqrt{\frac{\gamma}{2}} \Xi(t) \right] \cos(\omega_0 t). \quad (3.2)$$

The first term in this expression, $\langle X_\phi \rangle_c^I \equiv \text{Tr}_{\text{sys}}\{X_\phi \tilde{\rho}_c(t)\}$ is the slowly varying expectation value of the quadrature component

$$X_\phi \equiv ae^{i\phi} + a^\dagger e^{-i\phi} \quad (3.3)$$

(in the rotating frame). The second contribution in Eq. (3.2) is defined as

$$\Xi(t) \equiv \int_{-\infty}^t d\tau \cos(\omega_0 \tau + \phi) \tilde{Z}(t - \tau) \xi(\tau). \quad (3.4)$$

It describes the noise which passes through the feedback circuit. The stochastic mean of $\Xi(t)$ is zero due to the vanishing mean of the white noise variable $\xi(t)$, and the correlation function is given by

$$\langle \Xi(t) \Xi(t') \rangle \approx \delta_B(t - t') + O\left(\frac{B}{\omega_0}\right). \quad (3.5)$$

Here $\delta_B(t - t')$ denotes a delta-function for functions which vary on a slow time scale much larger than B^{-1} .

Thus for a clear separation of time scales,

$$G\gamma, \delta, \Gamma_{\text{eff}} \ll B \ll \omega_0, \nu_T, \quad (3.6)$$

the current given in Eq. (3.2) is proportional to the slowly varying expectation value of X_ϕ , and has a noise term which is delta-correlated on a time scale of the system evolution in the rotating frame.

B. Quantum feedback dynamics

The feedback current (3.2) for $\phi = -\pi/2$ is proportional to the slowly varying momentum of the particle. For the cooling of the ion motion, we apply a linear force which is proportional to the the feedback current (3.2). For a trapped ion, this can be realized by applying a voltage on the trap electrodes, which leads to a displacement of the trap center. The effect of the feedback force is given by the interaction picture Hamiltonian

$$H_{\text{fb}} = I_{\text{fb},c}(t - \tau) \tilde{z}_I(t). \quad (3.7)$$

In this equation, $\tilde{z}_I(t) \equiv U^\dagger \tilde{z} U$ is proportional to the position operator in the interaction picture, while τ denotes the finite time delay in the feedback loop, which we require to be much smaller than the trap frequency $\tau \ll 1/\nu_T$. The master equation (2.34) has to be complemented with the feedback term,

$$(S): \quad [d\tilde{\rho}_c(t)]_{\text{fb}} = I_{\text{fb},c}(t - \tau) (-i) [\tilde{z}_I(t), \tilde{\rho}_c(t)] dt, \quad (3.8)$$

which has to be interpreted as a *Stratonovich* stochastic dif-

ferential equation [7]. For the slow dynamics of the density matrix in the rotating frame, we can make a rotating wave approximation and neglect rapidly rotating terms $\sim \exp(\pm 2i\omega_0 t)$. The filtered noise (3.4) is delta-correlated on time scales slower than B^{-1} , thus we have the coarse grained evolution of the density matrix

$$(S): \quad [d\tilde{\rho}_c(t)]_{\text{fb}} = \frac{G}{2} \gamma \eta \langle X_\phi \rangle_c^I(t - \tau) dt \mathcal{K} \tilde{\rho}_c(t) + \frac{G}{2} \sqrt{\frac{\gamma}{2}} dW_\Xi(t - \tau) \mathcal{K} \tilde{\rho}_c(t), \quad (3.9)$$

with the feedback operator

$$\mathcal{K} \tilde{\rho}_c(t) \equiv -i [\tilde{z}, \tilde{\rho}_c(t)] \quad (3.10)$$

and the ‘‘slow’’ Wiener increment $dW_\Xi(t) \equiv \Xi(t) dt$.

The total evolution of the system is determined by the conditioned master equation (2.34) plus the contribution from the feedback loop (3.9). To combine the two equations, we have to convert Eq. (3.9) from Stratonovich to Ito form. The total conditioned evolution is

$$(I): \quad d\tilde{\rho}_c(t) = \tilde{\mathcal{L}} \tilde{\rho}_c + \sqrt{\frac{\gamma}{2}} \mathcal{H} dW(t) \tilde{\rho}_c(t) + \left(\frac{G}{2} \gamma \eta \langle X_\phi \rangle_c^I(t - \tau) dt + \frac{G^2}{16} \gamma \mathcal{K} dt + \frac{G}{2} \sqrt{\frac{\gamma}{2}} dW_\Xi(t - \tau) \right) \mathcal{K} \tilde{\rho}_c(t), \quad (3.11)$$

where

$$\tilde{\mathcal{L}} \tilde{\rho}_c \equiv \mathcal{L}_{\text{LC}} \tilde{\rho}_c - i [\delta a^\dagger a, \tilde{\rho}_c] \quad (3.12)$$

[cf. Eq. (2.25)] is the laser cooling Liouvillian in the rotating frame.

Because the exact photocurrent can not be kept track of in experiments, Eq. (3.11) is of limited use. The goal is to derive an equation for the ensemble averaged density operator. We follow the derivation given by Wiseman and Milburn in [7], where the measured current is fed back directly, and adopt it for our model. Assuming that the state at time $t - \tau$ and all previous times is known, we take the ensemble average $E[\cdot]$ of Eq. (3.11) over the trajectories in $(t - \tau, t]$. We then formally divide by dt and for convenience redefine $\rho(t) \equiv E[\tilde{\rho}_c(t)]$:

$$(I): \quad \dot{\rho}(t) = \tilde{\mathcal{L}} \rho(t) + \frac{G}{2} \gamma \eta \langle X_\phi \rangle_c^I(t - \tau) \mathcal{K} \rho(t) + \frac{G}{2} \sqrt{\frac{\gamma}{2}} \mathcal{K} E[\Xi(t - \tau) \tilde{\rho}_c(t)] + \frac{G^2}{16} \gamma \mathcal{K}^2 \rho(t). \quad (3.13)$$

The density matrix $\rho(t)$ is still conditioned on the evolution up to time $t - \tau$, but not conditioned on trajectories in $(t - \tau, t]$. The ensemble average $E[\langle X_\phi \rangle_c^I(t - \tau) \tilde{\rho}_c(t)]$ factorizes because $\rho_c(t - \tau)$ is assumed known. Under the Markov approximation, we let τ go to zero, while due to the coarse

graining of the time evolution in Eq. (3.8), dt will still be larger than this small delay. An expansion in τ yields

$$\begin{aligned}\tilde{\rho}_c(t) &= [1 + O(\tau)]\tilde{\rho}_c(t - \tau + dt) \\ &= [1 + O(\tau)] \left[1 + \sqrt{\frac{\gamma}{2}} dW(t - \tau) \mathcal{H} \right] \tilde{\rho}_c(t - \tau).\end{aligned}\quad (3.14)$$

We now can evaluate the remaining ensemble average in Eq. (3.13) because $dW(t - \tau)$ is stochastically independent from $\tilde{\rho}_c(t - \tau)$. We obtain

$$\begin{aligned}E[\Xi(t - \tau)\tilde{\rho}_c(t)] &= \sqrt{\gamma} \mathcal{H} E[\Xi(t - \tau)\xi(t - \tau)]\rho(t) \\ &= \sqrt{\frac{\gamma}{2}} \eta [X_\phi \rho(t) + \rho(t) X_\phi \\ &\quad - 2\langle X_\phi \rangle_c^\dagger(t - \tau)\rho(t)],\end{aligned}\quad (3.15)$$

and thus the term in the last line, a conditional expectation value, cancels with the second term on the right hand side of Eq. (3.13). In going from the first to the second line in Eq. (3.15) we have dropped terms $\sim \exp(\pm i\omega_0 t)$.

With this last step, we can finally evaluate Eq. (3.13) and write down the *quantum feedback master equation* for the motional degrees of freedom:

$$\dot{\rho} = \tilde{\mathcal{L}}\rho + \frac{G}{4} \gamma \eta \mathcal{K} (X_\phi \rho + \rho X_\phi) + \frac{G^2}{16} \gamma \mathcal{K}^2 \rho. \quad (3.16)$$

The first term on the right hand side $\tilde{\mathcal{L}}$ is the laser cooling Liouvillian (3.12) in the rotating frame. The second term with \mathcal{K} given in Eq. (3.10) in the master equation is the feedback term. It acts back on the system and is responsible for cooling if we choose the parameters δ and ϕ appropriately. The last term in the master equation is a diffusive term of the form of a double commutator.

IV. RESULTS

In the last section we have shown that for a separation of timescales $\delta, \Gamma_{\text{eff}} \ll B \ll \omega_0, \nu_T$ we obtain an unconditioned (nonselective) master equation for the motional density matrix in the rotating frame. By inserting the definitions of $\tilde{\mathcal{L}}$ and \mathcal{K} the master equation reads

$$\begin{aligned}\dot{\rho} &= -i\delta[a^\dagger a, \rho] + A_- \mathcal{D}[a] + A_+ \mathcal{D}[a^\dagger] - i\frac{G}{4} \gamma \eta [\tilde{z}, X_\phi \rho + \rho X_\phi] \\ &\quad - \frac{G^2}{16} \gamma [\tilde{z}, [\tilde{z}, \rho]].\end{aligned}\quad (4.1)$$

We have used the previously introduced variables $\tilde{z} = a + a^\dagger$ and $X_\phi = ae^{i\phi} + a^\dagger e^{-i\phi}$. In the first line of Eq. (4.1) we recover the master equation for laser cooling, with the corresponding heating and cooling rates A_\pm given in Eq. (2.23). The second line describes the effect of the feedback loop, where $\gamma = \varepsilon N \Gamma_{\text{eff}} / (1 + \alpha) \eta^2$ is the emission rate in the mirror mode and G is the gain parameter amplifying the feedback current. The first term in the second line depends on the phase shift ϕ

and as we will show below, leads to the expected damping for $\phi = -\pi/2$. The second term arises from the noise in the feedback current and leads to a momentum diffusion, i.e., heating.

To solve the feedback master equation (3.16), which is bilinear in the position and momentum \hat{z} and \hat{p}_z , it is convenient to use a Wigner function representation [5] of the density matrix. This gives rise to a Fokker-Planck equation [19] for the Wigner function $W(\bar{z}, \bar{p}, t)$, describing an Ornstein-Uhlenbeck process, with position and momentum variables \bar{z} and \bar{p} . The Gaussian Wigner function is uniquely determined by its first and second position and momentum moments, and we will use the notation

$$\langle \bar{z}^r \bar{p}^s \rangle_W \equiv \int d\bar{z} d\bar{p} \bar{z}^r \bar{p}^s W(\bar{z}, \bar{p}, t), \quad (4.2)$$

which equals the symmetric expectation value of the corresponding operators. The bilinearity of Eq. (3.16) with respect to position and momentum gives rise to a closed set of equations for the first and second moments of the Wigner function, which can thus be solved analytically.

We are interested in the motional energy of the ion $E = \hbar \nu_T (\langle a^\dagger a \rangle + 1/2)$, where the expectation value for the number operator can be read off from the second moments of the Wigner function:

$$\langle a^\dagger a \rangle \equiv \langle n \rangle = \langle \bar{z}^2 \rangle_W + \langle \bar{p}^2 \rangle_W - \frac{1}{2}. \quad (4.3)$$

We will calculate this quantity for different choices of parameters in the following subsections.

A. Cold damping

In this subsection we show results for $\phi = -\pi/2$ and $\delta = 0$, i.e., the center of the band pass filter is set exactly to the trap frequency. The number expectation value for the steady state in this case is given by

$$\langle n \rangle_{ss} = \frac{N + \frac{1}{2} \eta \tilde{\gamma} (2N - 1) G + \frac{1}{8} \tilde{\gamma} G^2}{1 + 2 \eta \tilde{\gamma} G}, \quad (4.4)$$

where we have set $\tilde{\gamma} \equiv \gamma / \Gamma$. Taking the gain $G = 0$ yields $\langle n \rangle_{ss} = N$, i.e., if we do not use the feedback current to influence the ion, the steady state occupation will be the one for standard laser cooling. The occupation number has a negative slope at $G = 0$, i.e., it decreases, and for $G \rightarrow \infty$ it diverges (note that in our model $G\gamma$ has to be smaller than B). Thus our theory predicts a minimum for the occupation number at a nonvanishing optimal gain,

$$\langle n \rangle_{\min} = \frac{4(2N - 1) \eta^2 \tilde{\gamma} - 1 + \sqrt{1 + 8(2N + 1) \eta^2 \tilde{\gamma}}}{16 \eta^2 \tilde{\gamma}}. \quad (4.5)$$

With increasing solid angle ε we collect more information about the motional state of the system and hence the minimum $\langle n \rangle_{ss}$ is expected to decrease, which is shown in Fig. 4. With increasing ε the optimal gain is decreasing, because the feedback noise term is growing with G^2 while the damping term is linear in G .

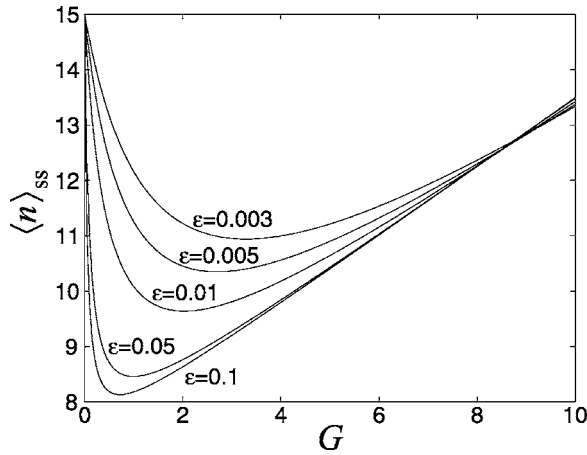


FIG. 4. Number expectation value for variable gain and solid angle fraction for $\eta=0.1$, $\phi=-\pi/2$, $\delta=0$, and $N=15$.

We show in Fig. 5 the decrease in the steady state phonon number with the gain. The relative decrease is larger with a higher laser cooling steady state phonon number N . For lower N , the mirror decay rate $\gamma \propto N$ is smaller and thus we get less information about the motional state of the atom, which limits the feedback cooling.

We will now expand $\langle n \rangle_{ss}$ in the limit of large ($N \gg 1$) occupation numbers. For a series expansion of (4.5) the formal expansion parameter is $N\sqrt{\epsilon}$, thus an expansion in the (usually also small) ϵ is only possible for very low N . We make an expansion for large N in the opposite limit (Doppler limit), while the condition $N\sqrt{\epsilon} \gg 1$ has to be satisfied. N can be tuned with, e.g., the laser detuning Δ_L . Then the minimal occupation number approximately reads

$$\langle n \rangle_{\min} = \frac{N}{2} + 4 \sqrt{\frac{1+\alpha}{\epsilon}} - \frac{1+\alpha}{N\epsilon}, \quad (4.6)$$

which implies that for a sufficiently large collection angle the minimal obtainable phonon number is above $N/2$ and thus

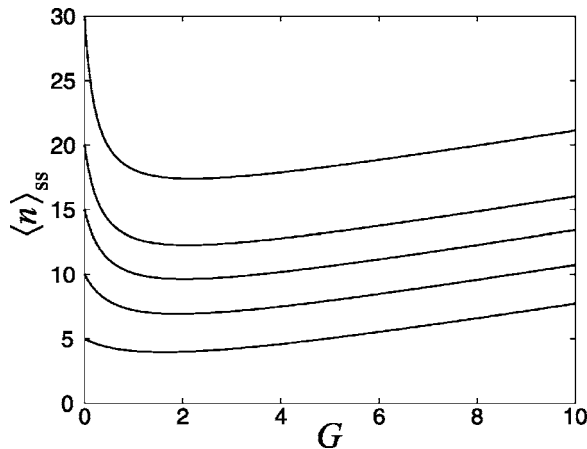


FIG. 5. Number expectation value for variable gain and steady state occupation number for $\eta=0.1$, $\phi=-\pi/2$, $\delta=0$, and $\epsilon=0.01$. N is given by the values at $G=0$ of the different curves, from bottom to top, $N=5, 10, 15, 20, 30$.

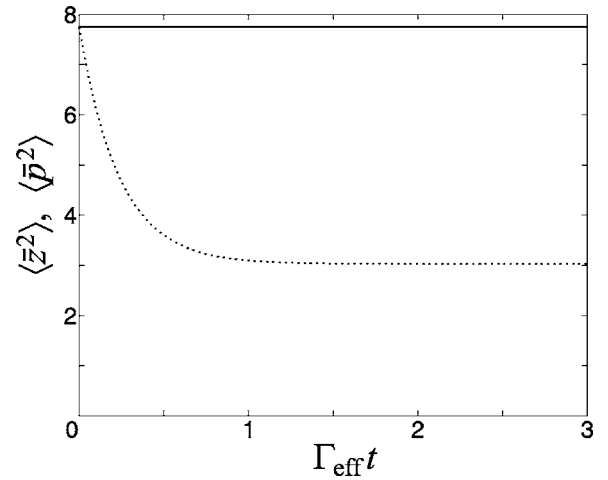


FIG. 6. Time evolution of the variances for $N=15$, $\tilde{\delta}=0$, $\epsilon=0.006$, $\eta=0.06$, $\phi=-\pi/2$, and $G \approx 1.51$. The solid line is the variance of the position $\langle \tilde{z}^2 \rangle_W$ and the dotted line is the variance of the momentum $\langle \tilde{p}^2 \rangle_W$.

feedback cooling alone cannot give a steady state. The reduction in the energy of the ion with time is due to the reduction in $\langle \tilde{p}^2 \rangle_W$, while $\langle \tilde{z}^2 \rangle_W$ is constant, as is shown in the time evolution in Fig. 6. Thus the Wigner function for the steady state will not be rotationally invariant, but “classically squeezed” in the momentum direction.

A phase space picture can demonstrate the action of the feedback on the system state [see Fig. 7(a)]. By feeding back a linear force f to the ion, we effectively apply a unitary operator of the form

$$U(t) \sim \exp(-ifxt). \quad (4.7)$$

This operator acts as a momentum kick on a state with a magnitude proportional to the momentum, which we have chosen by setting $\phi=-\pi/2$. The points in the Wigner function will tend towards the x axis, while the diffusion term will counteract the feedback term, leading to a steady state Wigner function.

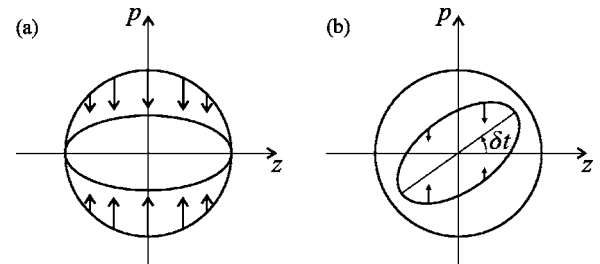


FIG. 7. Schematic drawing of the Wigner function error ellipses for the initial thermal state (circle) and the feedback-cooled state (ellipse). In (a), $\phi=-\pi/2$ and $\delta=0$, note that the position variance stays constant while the momentum variance is decreased. The action of the force is always a kick in the momentum direction and the force is proportional to the averaged momentum. In (b), $\delta \neq 0$ and $\phi \approx -\pi/2$, here the Wigner function is rotating and both variances are damped, resulting in lower energies.

The difference in the position and momentum variance can be quantified; we will give an expression for the amount of “squeezing,” i.e., the ratio between the two half-axis of the error ellipse for the Wigner function in phase space is obtained by rotating the axes of the ellipse:

$$r_\sigma \equiv \frac{\text{semiminor axis}}{\text{semimajor axis}} = \frac{1-g}{1+g}, \quad (4.8a)$$

$$g \equiv \frac{\sqrt{(\sigma_{zz} - \sigma_{pp})^2 + 4\sigma_{zp}^2}}{\sigma_{zz} + \sigma_{pp}}. \quad (4.8b)$$

Here $\sigma_{zz} = \langle \hat{z}^2 \rangle_W - \langle \hat{z} \rangle_W^2$ and $\sigma_{pp} = \langle \hat{p}^2 \rangle_W - \langle \hat{p} \rangle_W^2$ are the variances of position and momentum, respectively, and $\sigma_{zp} = \langle \hat{z}\hat{p} \rangle_W - \langle \hat{z} \rangle_W \langle \hat{p} \rangle_W$. As mentioned, due to the affection of only the σ_{pp} component, $\sigma_{zp} = 0$ in the case $\phi = -\pi/2$. The range of the squeezing parameter is $0 < r_\sigma \leq 1$, where a small value corresponds to strong squeezing and for $r_\sigma = 1$ the state is symmetric.

The time dependent Fokker-Planck equation is solvable analytically and the time scale of the cooling process is given by Γ_{eff} and $\Gamma_{\text{eff}} + 2\gamma\eta G$ corresponding to the usual Doppler cooling and the feedback cooling, respectively. This shows that the feedback cooling happens on a timescale faster than laser cooling alone.

B. Variable feedback phase

For a phase $\phi \neq -\pi/2$, the magnitude of the feedback force is proportional to the projection of the momentum on an other rotated axis in phase space. We have pointed out in Eq. (4.7) that the action of the linear force (shifted trap) is always a momentum kick. Thus the particle will always be “kicked too hard” or not hard enough towards the phase space center. We will calculate the regions of stability where the feedback can still lead to a steady state. Such a steady state will only form if $\Gamma_{\text{eff}} - 2G\gamma\eta \sin \phi > 0$ [19]. This is always true for negative angles. For positive angles $\phi > 0$, the gain has to fulfill the condition $G < \Gamma_{\text{eff}} / 2\gamma\eta \sin \phi$. If this condition is satisfied, a steady state number expectation value exists and reads

$$\begin{aligned} \langle n \rangle_{ss} &= [(1 - \eta\tilde{\gamma}G \sin \phi)(1 - 2\eta\tilde{\gamma}G \sin \phi)]^{-1} \\ &\times \left\{ N + \frac{1}{2}(4N - 1)\eta\tilde{\gamma}G \sin \phi + \frac{1}{8}\tilde{\gamma}G^2 \right. \\ &\times \left. [1 + 4\tilde{\gamma}\eta^2(2N + 1 - 2\sin^2 \phi)] - \eta\tilde{\gamma}^2 G^3 \sin \phi \right\}. \end{aligned} \quad (4.9)$$

From Eq. (4.9) we can see that an energy decrease via feedback cooling is only possible for angles $-\pi < \phi < 0$ by calculating the slope $\partial \langle n \rangle_{ss} / \partial G|_{G=0}$. Because Eq. (4.9) is of higher order in G than the equation we had for $\phi = -\pi/2$, (4.4), we will not give an analytical solution for the minimal gain and number occupation here. We also find that for $\phi \neq -\pi/2$ the optimal occupation number is higher than for $\phi = -\pi/2$ (compare related studies in [20]). The steady state occupation number for varying ϕ as a function of the gain is plotted in Fig. 8, where we can see that for nonoptimal phases the range of G for $\langle n \rangle_{ss} < N$ is shrinking.

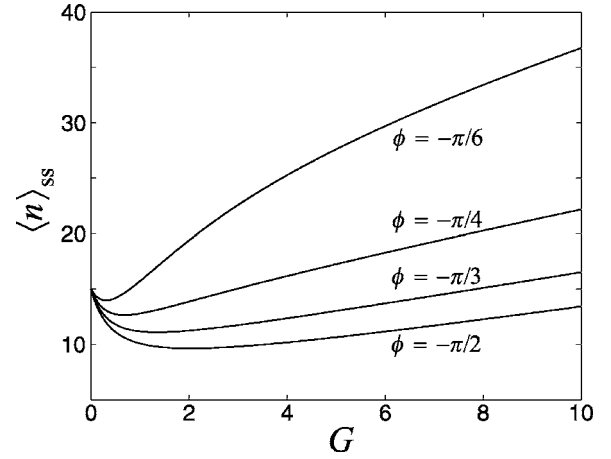


FIG. 8. Number expectation value for different feedback phases. Here $\delta=0$; thus the expectation values for $\phi \rightarrow \pi - \phi$ yield the same results. Other parameters: $N=15$, $\varepsilon=0.01$, and $\eta=0.1$.

For the special case of $\phi = \pi$ or $\phi = 0$, no cooling can be observed any more and the number expectation value is quadratic in G . In principle, a steady state with $\langle n \rangle_{ss} > N$ always exists with

$$\langle n \rangle_{ss} = N + \frac{1}{8}\tilde{\gamma}G^2[1 + 4\eta^2\tilde{\gamma}(2N + 1)]. \quad (4.10)$$

The more interesting feature of the $\phi = \pi$ case is that in the master equation (3.16) the feedback term (second term) reduces in a rotating wave approximation to a Hamiltonian term of the form $-i[\Delta\nu a^\dagger a, \rho]$. For this case we observe a small shift $\Delta\nu$ in the frequency of the trap linearly proportional to the gain. In this paper, we have not discussed the detailed experimental setup used to apply the force to the ion, which would be necessary for the knowledge of the exact forces acting on the ion. For $\phi = \pi$ one can measure the frequency shift in the location of the sideband and determine the conversion factor from the gain parameter G used in this paper and an experimental gain factor, which might be the real electronic gain in the feedback loop.

C. Rotation in phase space

We have shown that the phase $\phi = -\pi/2$ we chose leads to the lowest energy of the motional state of the ion. The variance for the position operator $\langle \hat{z}^2 \rangle_W = (2N + 1)/4$ remains constant with time as shown, e.g., in Fig. 6, thus posing a lower limit to the obtainable energy. The detuning δ of the local oscillator in the feedback loop from the trap frequency creates a tunable slow rotation of the (interaction picture) Wigner function in phase space. This results in “squeezing” of all quadrature components [see Fig. 7(b)], and the Wigner function can regain a symmetric shape. Of course the timescale for this rotation has to be much slower than the filter bandwidth B .

For the time evolution of the variances, the effect of the detuning is illustrated in Fig. 9. We see the time evolution of an initially thermal (symmetric) state with an occupation number of N . In contrast to Fig. 6 the width of the Wigner

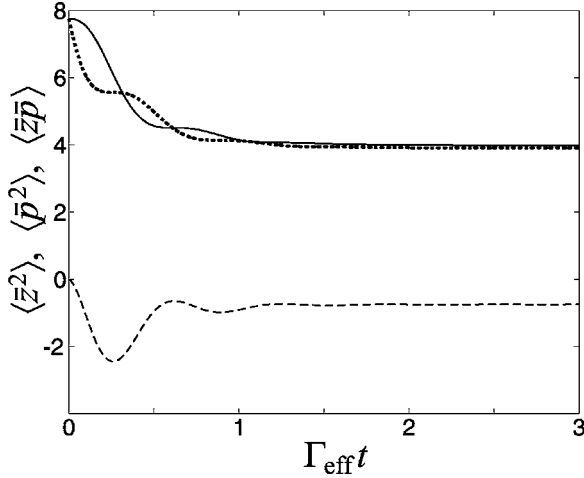


FIG. 9. Time evolution of variances $\langle \bar{z}^2 \rangle_W$, $\langle \bar{p}^2 \rangle_W$, and $\langle \bar{z}\bar{p} \rangle_W$ for $N=15$, $\tilde{\delta}=5$, $\varepsilon=0.006$, $\eta=0.06$, $\phi \approx -85^\circ$, and $G \approx 2.03$. The solid line is the variance of the position $\langle \bar{z}^2 \rangle_W$, the dotted line is the variance of the momentum $\langle \bar{p}^2 \rangle_W$, and the dashed line is $\langle \bar{z}\bar{p} \rangle_W$.

function in the momentum and the position space are alternately decreased until they reach the new feedback steady value. For a larger detunings the two variances decrease equally in time and energetically lower states can be reached.

For a rotation of the Wigner function with the frequency δ , we have to compare this rotation time scale with the cooling time scale γ . For γ comparable to δ the optimal phase is shifted with respect to $-\pi/2$ because the Wigner function is rotating in phase space during the cooling time. When the detuning is much larger than the cooling rate, the Wigner function ellipse direction will not be resolved during the cooling time and thus the optimal phase returns to $-\pi/2$. By numerical optimization (Fig. 10) we find that the optimal phase is shifted from $-\pi/2$ asymmetrically with respect to the detuning δ . It reaches its maximal excursion for a value of $\tilde{\delta} \equiv \delta/\Gamma_{\text{eff}} \approx 1$, for higher detunings the optimal phase approaches $-\pi/2$ again. For these optimal values, we plot in

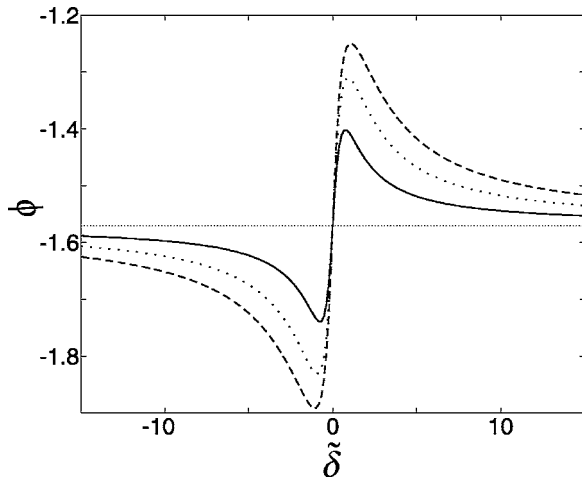


FIG. 10. Optimal phase as function of the detuning with $\eta=0.1$, $\varepsilon=0.006$, and $N=10$ (solid line), $N=17$ (dotted line), and $N=24$ (dashed line). The weak dotted line marks $\phi = -\pi/2$.

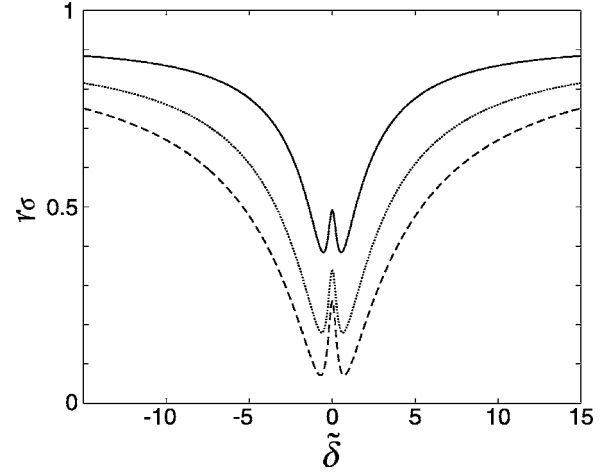


FIG. 11. Squeezing parameter r_σ for the optimal phase at a given detuning and the minimal energy state with $\varepsilon=0.006$, $\eta=0.06$ and with the variable parameter $N=10$ (solid line), $N=17$ (dotted line), and $N=24$ (dashed line).

Fig. 11 the squeezing parameter r_σ (4.8a), which is one for a symmetric Gaussian state. We see that the state at no detuning is “classically squeezed” as we already mentioned in Sec. IV A and the squeezing increases up to $\tilde{\delta} \approx 1$, then upon approaching $\tilde{\delta} \rightarrow \infty$ the squeezing parameter approaches one, and the state is thermal.

For increasing δ , we also show that the number expectation value is decreasing. We will not give an analytic expression for $\langle n \rangle_{ss}$ for an arbitrary δ here. We merely calculate the minimal number of phonons in the limit of large δ . For this we require an additional separation of the timescales between the effective feedback cooling rate and the detuning, while the other time scale inequalities still hold:

$$\gamma \ll \delta \ll B. \quad (4.11)$$

With these new conditions we take the detuning $\delta \rightarrow \infty$, where the optimal feedback phase is again $\phi = -\pi/2$, and get for the occupation number:

$$\langle n \rangle_{ss} = \frac{N - \frac{1}{2} \eta \tilde{\gamma} G + \frac{1}{8} \tilde{\gamma} G^2}{1 + \eta \tilde{\gamma} G}. \quad (4.12)$$

The minimal occupation number for the same limit we took in deriving Eq. (4.6) we get for $N \gg 1$:

$$\langle n \rangle_{\min} \approx \sqrt{\frac{1 + \alpha}{2\varepsilon}} - \frac{1}{2} - \frac{2(1 + \alpha)}{8\varepsilon N}. \quad (4.13)$$

This expression does not include the large term $N/2$ any more and thus the obtainable energy for large N has an upper bound which is independent of N , thus feedback cooling alone can give a thermal (symmetric) state with a temperature below the Doppler temperature.

V. CONCLUSION

In this paper we have studied quantum feedback cooling of a trapped ion in front of a mirror. This work is motivated

by recent experiments [11], and—as shown in [12]—provides a quantitative understanding of the experimental results.

In the setup discussed in this paper the final temperatures are limited by the collection efficiency, ε , and the constant scattering of photons for the position measurement. This combination of heating due to the recoil, and laser cooling due to the red detuning of the laser leads to a steady state temperature (Doppler limit). The effect of quantum feedback cooling is studied as an additional cooling mechanism on top of the ongoing laser cooling. For the experimentally relevant parameters this leads to sub-Doppler cooling, but it seems difficult to achieve ground state cooling in the trap along these lines. As shown in a parallel publication [14], we can devise a purely *dispersive* and thus noninvasive readout of the velocity of the trapped ion based on the variation of the index of refraction with velocity, i.e., based on electromagnetically induced transparency. Such a scheme allows, under idealized conditions, ground state cooling of the ion purely by quantum feedback.

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APPENDIX A: ADIABATIC ELIMINATION, LAMB-DICKE LIMIT AND LASER COOLING

This appendix fills in the details of deriving the QSME (2.17) from the QSSE (2.10) under the assumption of weak driving and small Lamb-Dicke parameter. Note that we will need to consider two different Lamb-Dicke parameters due to the exciting laser which is not collinear with the z axis. As in Sec. II A we denote $\tilde{\eta} \equiv \eta \sin \chi$. Inserting the ansatz (2.15) into the QSSE (2.10) and transforming to an interaction picture with respect to H_T we get

$$|\psi_e(t)\rangle = \frac{i\Omega}{2} \left(\frac{1 - \frac{1}{2}\tilde{\eta}^2 a^\dagger a}{-i\Delta_L + \frac{\Gamma}{2}} + \frac{i\tilde{\eta} a e^{-i\nu_T t}}{-i(\Delta_L - \nu_T) + \frac{\Gamma}{2}} + \frac{i\tilde{\eta} a^\dagger e^{i\nu_T t}}{-i(\Delta_L + \nu_T) + \frac{\Gamma}{2}} \right) |\psi_g(t)\rangle. \quad (\text{A1})$$

We insert this expression back into (2.10). We obtain

$$d|\psi_g(t)\rangle = \left\{ -\frac{\Omega^2}{4} \left[\frac{1}{-i\Delta_L + \frac{\Gamma}{2}} + O(e^{\pm 2i\nu_T t}) + \frac{\tilde{\eta}^2 a^\dagger a}{-i(\Delta_L - \nu_T) + \frac{\Gamma}{2}} + \frac{\tilde{\eta}^2 a a^\dagger}{-i(\Delta_L + \nu_T) + \frac{\Gamma}{2}} \right] dt + \frac{i\Omega}{2} \left[\frac{1}{-i\Delta_L + \frac{\Gamma}{2}} dC_1^\dagger - \left(\frac{i\tilde{\eta} a e^{-i\nu_T t}}{i(\Delta_L - \nu_T) - \frac{\Gamma}{2}} + \frac{i\tilde{\eta} a^\dagger e^{i\nu_T t}}{i(\Delta_L + \nu_T) - \frac{\Gamma}{2}} \right) dC_1^\dagger + \frac{\eta(a e^{-i\nu_T t} + a^\dagger e^{i\nu_T t})}{-i\Delta_L + \frac{\Gamma}{2}} dC_2^\dagger \right] \right\} |\psi_g(t)\rangle \quad (\text{I}) \quad (\text{A2})$$

with

$$dC_1^\dagger \equiv \sqrt{\Gamma_b} \int du \sqrt{N(u)} dB_u^\dagger + \sqrt{\Gamma_m} \sin(k_{eg}L) dB_m^\dagger, \quad (\text{A3})$$

$$dC_2^\dagger \equiv -i\sqrt{\Gamma_b} \int du \sqrt{N(u)} u dB_u^\dagger + \sqrt{\Gamma_m} \cos(k_{eg}L) dB_m^\dagger. \quad (\text{A4})$$

Consistent with the above approximations we neglect here and in the following terms oscillating at twice the trap frequency ν_T . Physically speaking, the fourth line of Eq. (A2) will correspond together with third line to a heating and cooling term, and the last line describes a diffusive term (cf. Fig. 2).

Taking the trace over the background modes to define a reduced density operator $\hat{w}(t)$ according to (2.16) we use the Ito rules, e.g.,

$$\text{Tr}_b\{dB_u^\dagger(t)|\psi_g(t)\rangle\langle\psi_g(t)|dB_{u'}(t)\} = \delta(u - u')\rho(t)dt$$

to derive Eq. (2.17).

APPENDIX B: HOMODYNE PHOTODETECTION AND THE DIFFUSION APPROXIMATION

As we have seen in Sec. II F, the statistics of the detected photons in the mirror mode are determined by the Poissonian stochastic variable $dN_c(t)$. Like in homodyne detection, where a strong local oscillator beats with the photodetection signal from a quantum system, an elastic scattering term beats with the signal given by the coupling of the light to the ion's motion [cf. Eq. (2.31)]. The parameter which gives the difference in the magnitudes of these terms is the Lamb-Dicke parameter η . The stochastic properties of $dY_c(t)$ defined in Eq. (2.32) are given by

$$\langle dY_c(t) \rangle = \gamma(\vec{z})_c(t)dt, \quad (\text{B1})$$

$$dY_c^2(t) = \left(\frac{dN_c(t) - \frac{1}{2}\gamma dt}{\eta} \right)^2 \xrightarrow{\eta \ll 1} \frac{\gamma}{2\eta^2} dt. \quad (\text{B2})$$

This implicates that the stochastic variable has Gaussian properties, and is associated with a white noise probability distribution. Thus $dY_c(t) = \sqrt{\gamma/2}/\eta dW(t)$ where $dW(t)$ is a Wiener increment.

The evolution of the system conditioned on measuring the photocurrent can be seen by expanding the first bracket in

the stochastic master equation (2.30) to first order in the Lamb-Dicke parameter η , and noting that the second bracket in (2.30) is

$$dN_c(t) - \langle dN_c(t) \rangle = \sqrt{\gamma/2} dW(t). \quad (\text{B3})$$

Thus, using the formal derivative $\xi(t) = dW(t)/dt$, we obtain the conditioned equation for the reduced density matrix, Eq. (2.34).

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