

Quantum Hall Physics with Cold Atoms in Cylindrical Optical Lattices

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We propose and study various realizations of a Hofstadter-Hubbard model on a cylinder geometry with fermionic cold atoms in optical lattices. The cylindrical optical lattice is created by counter-propagating Laguerre-Gauss beams, i.e. light beams carrying orbital angular momentum. By strong focusing of the light beams we create a real space optical lattice in the form of rings, which are offset in energy. A second set of Laguerre-Gauss beams then induces a Raman-hopping between these rings, imprinting phases corresponding to a synthetic magnetic field (artificial gauge field). In addition, by rotating the lattice potential, we achieve a slowly varying flux through the hole of the cylinder, which allows us to probe the Hall response of the system as a realization of Laughlin's thought experiment. We study how in the presence of interactions fractional quantum Hall physics could be observed in this setup.

Introduction. – For our understanding of quantum many-body systems, considering spherical, cylindrical, toroidal, or even more exotic geometries has in many situations proven to be of key importance [1]. This is because phenomenologically distinct physical properties may be revealed by imposing various boundary conditions. Prominent examples along these lines include protected edge states [2], topological ground state degeneracies [3], and spectral flow in response to fluxes threading the holes of a system [4]. The natural question, as to what extent such theoretically intriguing constructions can become experimentally viable, can be seen as a challenge in quantum engineering. For cold atoms in optical lattices [5, 6] with planar geometries, remarkable progress has been reported in devising [7–13] and experimentally probing [14–18] gauge fields and topological states [19, 20]. It is the purpose of the present work to present a microscopic model for a *cylindrical optical lattice* in real space that realizes the Hofstadter model [21] of fermionic atoms subject to a perpendicular synthetic magnetic field [see Fig. 1a)]. Our proposal builds on, and is motivated by advances to generate orbital angular momentum (OAM) light beams [22, 23]. The simulation of artificial magnetic fields on non-trivial geometries has also been addressed with the experimentally intriguing idea of a *synthetic dimension* [24–28], i.e. a manifold of internal states coupled by Raman lasers that are interpreted as (a small number of) lattice sites. In contrast, our work focuses on *real space* cylindrical lattices, which becomes essential once interactions are included to realize fractional Quantum Hall (FQH) physics [29–31].

For non-interacting atoms we present a microscopic realization of the fermionic Hofstadter Hamiltonian [21] on a cylinder

$$H_0 = -\sum_{\mathbf{j}} J_{\mathbf{j}}^z e^{i\varphi_{\mathbf{j}}} a_{\mathbf{j}}^\dagger a_{\mathbf{j}+\hat{z}} - \sum_{\mathbf{j}} J_{\mathbf{j}}^\phi a_{\mathbf{j}}^\dagger a_{\mathbf{j}+\hat{\phi}_{\mathbf{j}}} + \text{h.c.}, \quad (1)$$

where $J_{\mathbf{j}}^z$ and $J_{\mathbf{j}}^\phi$ denote amplitudes for nearest neighbor

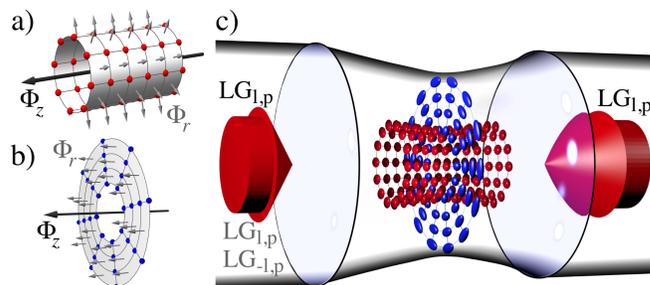


FIG. 1. (Color online) Lattices with real cylindrical (spanned by red points) and ring shaped (blue points) geometries in the focal plane of a double lens system. Independent threading of artificial magnetic fluxes through the surface or the interior of the cylinder [panels a) and b)] is possible in both scenarios. Panel c) shows the general structure and geometry of the proposed setup. The lattice is located in the focus of tightly focused lasers carrying orbital angular momentum.

hopping to lattice site \mathbf{j} by a displacement vector \hat{z} and $\hat{\phi}_{\mathbf{j}}$ in axial and azimuthal direction, respectively, $a_{\mathbf{j}}$ annihilates a spinless fermion at site \mathbf{j} , and the spatially dependent phase factors $e^{i\varphi_{\mathbf{j}}}$ affect the neutral atoms analogous to the effect of a magnetic field on charged particles. Adding interactions, the total Hubbard Hamiltonian is given by $H = H_0 + H_I$. For spinless fermionic atoms the natural (minimal) interaction is the nearest-neighbor interaction $H_I = U \sum_{\langle i,j \rangle} n_i n_j$, where $n_i = a_i^\dagger a_i$ is the particle number on site i .

Below, we address two main questions. First, we detail how a cylindrical optical lattice threaded by a synthetic magnetic flux Φ_r field can be created. This is achieved by interfering tightly focused red-detuned laser beams and Raman beams that carry OAM [see Fig. 1c) for a schematic]. In addition, by rotating the cylindrical lattice potential around its axis, we are capable of mimicking a slowly varying flux Φ_z threading the cylinder in the

axial direction [see Fig. 1a)] (see also Ref. [32]). We argue how the resulting spectral flow hallmarking can be experimentally observed. These ingredients allow us to implement Laughlin's original thought experiment [4], explaining the integer quantum Hall effect [4, 33, 34] in terms of spectral flow.

Second, going beyond the single particle picture, we investigate numerically the effect of local interactions and outline how FQH physics [29–31] can occur in our cylindrical model. We compare the Hofstadter-Hubbard model with real space cylinder geometry proposed here to its counterpart involving a synthetic dimension [28]. Including interactions, synthetic and real dimensions are shown to lead to a qualitatively different behavior. The physical reason behind this is that spatial locality of many-body interactions and localized wave functions giving rise to topological protection occur naturally in real samples but may be non-generic or require fine tuning in the context of synthetic dimensions.

Cylindrical optical lattices. – We are interested in creating optical lattices with annulus or cylinder geometry as depicted in Figs. 1b) and 1c). Moreover, building on Raman assisted tunneling techniques [7, 9, 10] and a rotation of the lattice potential, respectively, we aim to engineer two kinds of artificial magnetic fields. One corresponds to a magnetic flux Φ_r piercing the surface of the lattice [7, 14, 21], the second one gives rise to a flux Φ_z threading the hole of the system in axial direction [4]. The spinless fermionic atoms are assumed to occupy a single Zeeman m -state, where the quantization axis is defined by homogeneous magnetic field $\vec{B} = B_0 \hat{z}$. The key ingredients for our proposal are tightly focused light beams carrying OAM [35–37]. A large focus angle is essential to reach lattice spacings d on the order of the optical wavelength $\lambda \equiv 2\pi/k$ and thus sufficiently large energy scales $E_l = \frac{\hbar^2 \pi^2}{2md^2}$. Some basic features, however, can be understood in terms of a paraxial description valid for weakly focused light. There, Laguerre Gaussian (LG) laser beams $LG_{l,p}$ are conveniently employed to describe light carrying an OAM of $l\hbar$ per photon [38], which is reflected in an azimuthal phase dependence $e^{il\phi}$ of the electric field amplitude. In this framework, an intensity modulation $I \sim \cos^2(l\phi)$ around the beam axis arises naturally when two beams $LG_{l,p}$ and $LG_{-l,p}$ are interfered [39–41]. Together with the radial dependence of the intensity, it gives rise to lattice potentials consisting of $p+1$ concentric rings with $2l$ azimuthal sites each [see Fig. 2a-b)]. Beams with $l = 25$ and $p = 4$ are experimental state of the art [42]. The paraxial approximation would limit the lattice constant d to values much larger than λ [43]. This limitation stands in stark contrast to the key requirements of reaching lattice constants $d \sim \lambda/2$ in the interest of experimentally viable energy scales. Therefore, we here investigate the optical potential generated by *tightly focused* beams with OAM, thus going conceptually beyond the simplified picture drawn by the paraxial approximation [see Fig. 1c)]. The dependence of d on

the focusing angle θ [37] is displayed in Fig. 2c), showing that $d \approx \lambda/2$ is achievable in the non-paraxial regime of this more general approach.

Strong focusing might be achieved using different means. Here, for concreteness, we consider two aplanatic lenses with focal length f to focus incident LG beams propagating along the z -axis, i.e., the optical axis of our setup [see Fig. 1 c)]. For an incident LG beam with angular momentum l and circular polarization $\mathbf{e}_+ = -(\mathbf{e}_x + i\mathbf{e}_y)/\sqrt{2}$, the electric field \mathbf{E}_l close to the focal plane is given by [44, 45]

$$\mathbf{E}_l = e^{il\phi} \left(\mathcal{E}_{|l|}^+ \mathbf{e}_+ + \mathcal{E}_{|l+1|}^z e^{i\phi} \mathbf{e}_z - \mathcal{E}_{|l+2|}^- e^{2i\phi} \mathbf{e}_- \right). \quad (2)$$

We note that in this non-paraxial regime the polarization can no longer be separated from the spatial mode profile. Moreover, the focused field is no longer transverse and the different polarization amplitudes are given by $\mathcal{E}_l^\sigma(\rho, z) = E_0 \int_0^{\theta_m} d\theta \sin\theta g_l(\theta) \sqrt{\cos\theta} h_\sigma(\theta) J_l(k\rho \sin\theta) e^{ikz \cos\theta}$, where we abbreviated $h_\pm(\theta) = 1 \pm \cos\theta$ and $h_z(\theta) = -\sqrt{2}i \sin\theta$, as well as $g_l(\theta) = \xi^{|l|} L_p^{||}(\xi^2) e^{\xi^2/2}$, with $\xi = \sqrt{2} \frac{f}{w_0} \sin\theta$. The parameters E_0 and w_0 specify the intensity and the waist of the beams incident on the lens. The numerical aperture of the lens enters via $NA = \sin\theta_m$. The Bessel functions and the associated Laguerre polynomials are denoted by J_l and L_p^l , respectively.

Superposing two such lasers with the same propagation direction but with opposite l , we obtain an intensity pattern $I_\sigma(\rho, \phi, z)$, which is invariant by rotation of π/l around the optical axis [see Fig.2a-b)] in each polarization component I_σ . Remarkably, even though the azimuthal phase dependence of the electric field is not simply $e^{il\phi}$ [cf. Eq. (2)] like in the paraxial treatment, this exact symmetry is guaranteed due to the circular polarization of the incident light. For details we refer to the supplemental material. The optical potential is then given by $V(\vec{r}) = \sum_{\sigma \in \{+, -, \pi\}} \alpha_\sigma I_\sigma(\rho, \phi, z)$, where α_σ is the atomic polarizability. For the sake of simplicity we use $\alpha_\sigma \equiv \alpha < 0$ in what follows.

The above azimuthal lattice structure allows us to realize two distinct geometries including artificial magnetic fields. One possibility is to generate an *annulus geometry* by confining atoms with an additional standing wave laser beam along z to a *single disk* in the focal plane [Fig. 1b), blue sites in Fig. 1c)], consisting of a series of concentric ring lattices, with $d \sim \lambda/2$. In the radial direction, the rings are decoupled by an energy offset stemming from the radially varying laser intensity. Such an offset can be used to engineer the synthetic gauge field Φ_r , as we detail below for the cylinder geometry, which is the main subject of our subsequent analysis.

To realize the cylinder geometry, we introduce a second pair of lasers in a standing wave configuration using the same lensing system. Its purpose is to create a lattice structure in z -direction and to enhance the mentioned

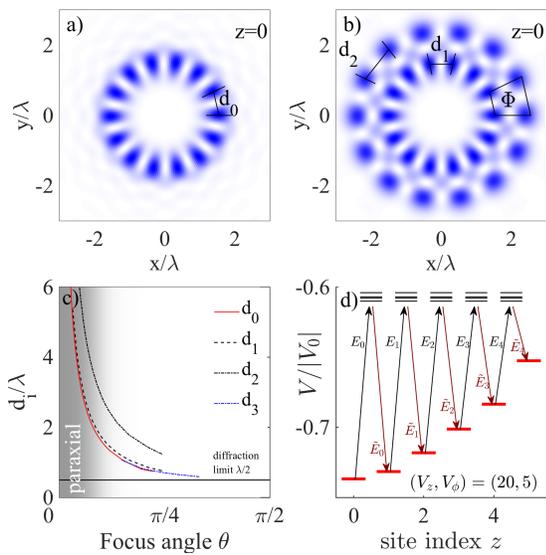


FIG. 2. (Color online) Panels a), b) show the total intensity pattern of strongly focused, nonparaxial beams with angular momentum $l = 7$ for $p = 0$ and $p = 1$, respectively. Panel c): Scaling of intersite distances d_i [see a) and b)] with the focus angle and its convergence to the diffraction limit for strong focusing, away from the grey paraxial regime. The distance d_3 refers to innermost ring for $l = 60$ and $p = 15$. Panel d) shows the single particle energy levels for the cylinder geometry (red lines) spanned like in Fig. 1a). The arrows show a Raman assisted axial hopping scheme.

radial energy barriers thus selecting the innermost ring of every disc [see red sites in Fig. 1c)]. In z -direction, these form a staircase of rings [see Fig. 1d)], where the steps are energy offsets due to decreasing light intensity away from the focal plane. This dynamically decouples the rings in z -direction. However, using Raman beams, a hopping with an imprinted phase pattern can be restored, as we discuss in detail below. Compared to the annulus geometry, this setup has the advantage that the azimuthal tunneling rates are more homogeneous. We hence focus in the following on this cylinder geometry.

On a single particle level we derive a tight binding model for our present potential $V(\vec{r})$ which is not fully separable, unlike simple cubic potentials. Assuming that the axial lattice depth V_z dominates the azimuthal depth V_ϕ , one may solve separately for the axial z -direction while in the $x - y$ plane requires a fully two-dimensional treatment, in order to calculate the azimuthal hopping matrix elements J_j^ϕ . Away from the focal plane, the decrease of the intensity [see Fig. 2b)] and of the lattice depth dominate over the increase of the lattice constant. Hence, the tunneling elements in azimuthal direction increase. For the parameters used in Fig. 3, the hopping amplitude varies by a factor of 1.25 (1.05) between the ring in the focal plane and 10-th (5-th) nearest neighbours.

Artificial magnetic field on a cylinder. – To mimic the magnetic flux Φ_r threading the surface of the cylin-

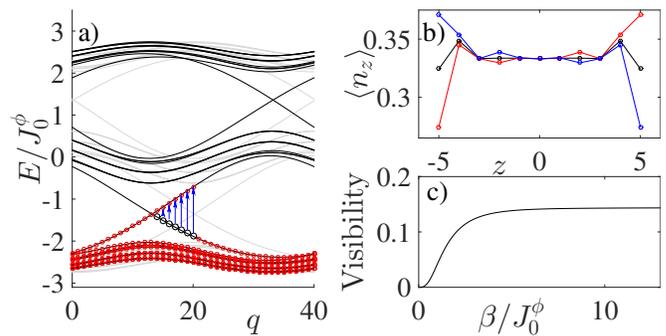


FIG. 3. (Color online) Laughlin experiment in cylindrical potential. Maximal azimuthal lattice height is $V_\phi = 5E_r$, for longitudinal $-V_z = 20E_r$ with the recoil energy $E_r = \frac{\hbar^2 k^2}{2m}$. Panel (a) shows the quasienergy spectrum of (1) shifted in quasimomentum space by artificial magnetic field threading flux $\Phi_{z,f} = 14\pi$. Gray curves show the spectrum for no magnetic field. Panel (b) shows $\langle n_i^\Phi \rangle$ — the average density of the gas after artificial probe magnetic field with flux Φ has been threaded (black — $\Phi_z = 0$, red — $\Phi_{z,f} = 14\pi$, blue — $\Phi_{z,f} = -14\pi$). Panel (c) shows effects of temperature on the predicted visibility $v = (\langle n_5^{\Phi=\Phi_{z,f}} \rangle - \langle n_5^{\Phi=\Phi_0} \rangle) / \langle n_5^{\Phi=\Phi_0} \rangle$.

der, we here adapt a Raman assisted tunneling scheme [7, 9, 10] to the cylindrical optical lattice. Conventionally, this scheme consists of two steps. First, the optical lattice potential is tilted in one spatial direction, which renders the hopping in this direction off-resonant. Second, two running Raman lasers are employed to put hopping assisted by a two-photon process back on resonance. This allows to imprint a spatially dependent phase pattern, using Raman beams with different wave vectors. In our present setup, the first step comes naturally due to the focusing as discussed above. The counterpart of the second step in the cylindrical geometry is to use Raman lasers carrying different OAM in axial direction. For Raman lasers carrying OAM l_a the hopping amplitude between sites \mathbf{j} and $\mathbf{j} + \hat{z}$ acquires the phase dependence

$$J_j^z = J_j^z e^{i(l_2 - l_1)\phi_j} = J_j^z e^{i\varphi_j}, \quad (3)$$

where $J_j^z \in \mathbb{R}$ and ϕ_j is the azimuthal angle of the lattice site \mathbf{j} . An atom hopping around a plaquette of the lattice with $2l$ azimuthal sites now picks up a nonzero phase $\delta\phi = 2s\pi/l$, ($s \equiv l_2 - l_1$). We note that the Raman lasers are focused via the same lensing scheme and thus the electric field is also of the form (2). Hence, different polarization components carry different phase dependencies. However it is easy to see that these non-paraxial effects do not change the phase pattern (3) imprinted on the hopping elements. The gradient of the onsite energies in axial direction can be made constant to a fair approximation by external potentials, e.g., by weakly focused laser field from the perpendicular direction. This allows to couple all sites using only a single pair of Raman lasers. The final tight-binding model is then of the form of the Hofstadter Hamiltonian on a cylinder (1). Its band structure may be probed using spectroscopy techniques

together with the time-of flight imaging (see Supplementary material).

Flux threading and Hall response. - The Hall response of a cylindrical system can be probed along the lines of Laughlin's famous thought experiment [4], where a small electric field in azimuthal direction is imposed on a gas of electrons by adiabatically threading a flux in axial direction through the cylinder. A quantum Hall system such as our Hofstadter model (1) then responds with a current in axial direction that is generated by spectral flow. The integer number of particles transported between the ends of the cylinder per threaded flux quantum then equals the quantized Hall conductance in units of e^2/h .

Our present setup offers a natural way to realize such a scenario. An artificial time dependent flux may be implemented by a slowly accelerated rotation with frequency $\Omega(t)$ of the cylindrical lattice around its symmetry axis, which translates the spectrum in quasimomentum by $\Delta k(t) = 2m\Omega(t)r^2/\hbar$ close to the focal plane. Experimentally, such a rotation is readily achieved by a frequency detuning of the counter-propagating laser beams generating the lattice potential. At the level of the tight-binding model (1), it imprints a complex phase to the azimuthal hopping, i.e., $J_j^\phi \rightarrow J_j^\phi e^{i\Delta k(t)}$. The small detunings caused by the rotation do not affect the other building blocks of our proposal. In Fig. 3b), we show the Hall response to such an axial flux of a system of ultracold fermions with magnetic flux $\Phi_r = 2\pi/3$ per plaquette [see Eq. (1)] at zero temperature and $1/3$ filling of the lattice, i.e., with one filled Chern band [see Fig. 3a)]. Depending on the orientation of the magnetic flux, atoms are transferred in different directions between the edges of the cylinder. For finite temperatures, the visibility of this charge pumping is shown in Fig. 3c).

Effect of interactions and fractional filling. - We now turn to the role of interactions, focusing on the possibility of realizing fractional quantum Hall (FQH) physics [29–31] on a cylinder. We are particularly interested in small cylindrical systems that may be realized in a first generation of experiments. Using exact diagonalization, we assess which signatures of FQH physics could be observable. We focus on one-third filling of the lowest band at flux $\Phi_r = 2\pi/3$. In this situation, short-ranged interactions are expected to stabilize FQH states [46, 47], more specifically the $1/3$ -Laughlin state [30, 46, 48]. In our model study we assume a nearest neighbor interaction between the spinless fermions $H_I = U \sum_{\langle i,j \rangle} n_i n_j$ [49], as naturally realized microscopically by magnetic interactions in dipolar Er and Dy fermionic quantum gases [50]. We compare our results for the *real space cylinder* to its counterpart involving a *synthetic dimension* [24]. While the free Hamiltonians are the same and give rise to a metallic phase, the interaction in the presence of a synthetic dimension is naturally of *infinite range* [51–53] along the synthetic dimension, since different lattice sites are not spatially separated in real space, i.e.

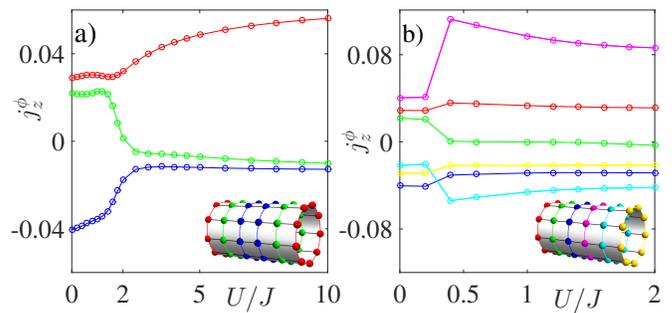
$$H_{\text{syn}} = \frac{U}{2} \sum_{i=0}^{2l+1} \left(\sum_z n_{i\hat{\phi}+z\hat{z}} \right)^2.$$


FIG. 4. (Color online) Azimuthal current j_z^ϕ around the rings of the cylinder as a function of the (repulsive) interaction strength for the real cylinder a) and the synthetic dimension analog b). We consider 6 fermions on a cylinder made of 6 rings with 9 sites each. Panel a) shows only the first three rings due to parity symmetry in z -direction.

In the cylinder geometry, a simple probe for a FQH state is to look at the interaction-induced emergence of chiral edge currents. To this end, we compute the value of the azimuthal current $j_z^\phi = \langle i e^{i\varphi_i} a_{j+\hat{\phi}_j}^\dagger a_j + \text{h.c.} \rangle$ in the groundstate on the different rings z as a function of interaction strength. As the system is translation invariant in the azimuthal direction, the current only depends on z . Our numerical results are shown in Fig. 4. For the real cylinder geometry [see Fig. 4a)], the groundstate remains in the zero momentum sector as expected for an incompressible liquid such as the Laughlin state. At small interaction, the current of the groundstate with local interactions is of the same order on each ring of the cylinder. With increasing interaction strength, the current on the edge increases while the current in the bulk decreases, resulting in a ratio between the two currents of around 5. While, given the small system sizes, the system does not display a fully insulating bulk yet, this may be seen as a first signature of FQH physics that could be observed in experiments on very small cylinders. Experimentally going beyond numerically accessible system sizes, this signature is expected to become increasingly clear. By contrast, the groundstate of the system involving a synthetic dimension [see Fig. 4b)] has a non-zero momentum for $U \sim 0.4J$ and becomes 2-fold degenerate reflecting spontaneous symmetry breaking.

Outlook. - The present proposal can be extended to create an optical lattice with torus geometry: this can be achieved with two concentric cylinders which, in our setup, are naturally decoupled due to a radial energy shift. Coupling of the cylinders at the edges can be restored by photon-assisted tunnelling, effectively sewing together the patterns to a seamless torus.

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I. SUPPLEMENTARY MATERIAL

In this Supplementary Material, we present more detailed quantitative analysis of the optical potential. Moreover, we microscopically determine the relevant interaction parameters for a realistic cylindrical lattice potential.

II. QUANTITATIVE PROPERTIES OF THE POTENTIAL

The interference pattern defining the lattice structure in the azimuthal direction is created by interfering two tightly focused beams that initially carry opposite orbital angular momentum (OAM) $l_1 = l, l_2 = -l$. For the specific example presented in this work we have chosen the Laguerre-Gaussian modes incident on the lensing system to carry $l = 60$ and the radial parameter $p = 15$. The focusing of the beam is described by the ratio of the focal length and the paraxial waist of a pre-lens beam, which here equals to $f/w_0 = 11$. The asymptotic propagation angle of the outermost intensity maximum determines the focus angle of the whole beam to the value of $\theta = 51.6^\circ$.

As discussed in the main text, the total intensity $I_t \sim |E_l + E_{-l}|^2$ splits into polarization components: $I = \sum_{\sigma \in \{+, -, \pi\}} I_\sigma(\rho, \phi, z)$, where each of them is of the form

$$I_\sigma(\rho, \phi, z) = a_\sigma(\rho, z) + b_\sigma(\rho, z) \cos^2(l\phi + c_\sigma(\rho, z)), \quad (4)$$

such that $I_\sigma(\rho, \phi, z) = I_\sigma(\rho, \phi + \pi/l, z)$. The total intensity $I_t = \sum_\sigma I_\sigma$ satisfies the same periodicity condition and can be expressed as:

$$I_t(\rho, \phi, z) = a_t(\rho, z) + b_t(\rho, z) \cos^2(l\phi + c_t(\rho, z)). \quad (5)$$

In the ideal paraxial case [1], the polarization of the electric field is not affected by the focusing, and in general $a_\nu = c_\nu = 0, \forall \nu \in \mathbb{R}^3$. However, in the nonparaxial regime the latter no longer holds. The most important parameters of the resulting optical potential are the values of a_t, b_t, c_t in the proximity of intensity maxima. We find (See Fig. 5) that in general the parameter c is close to zero, near the focal plane.

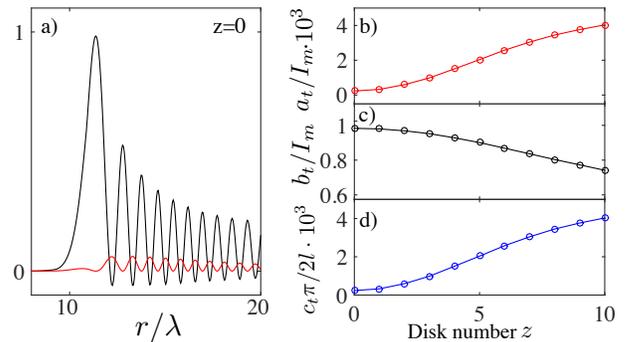


FIG. 5. (Color online) Properties of a pair of nonparaxial, tightly focused beam creating the azimuthal lattice described by parameters a_t, b_t, c_t from Eq. 4. Panel a) shows a_t, b_t computed along the r direction for $z = 0$. Panels b), c), d) show the a_t, b_t, c_t (black, red, blue respectively) coefficients at main intensity maxima for the innermost intensity ring as a function disk number z . a_t, b_t coefficients have been normalized by a maximum intensity I_m .

Remarkably, in the case of interfering copropagating beams with opposite initial OAM, the azimuthal oscillation amplitude b_t in the nonparaxial regime changes sign in the narrow region between subsequent maxima (see Fig. 4). In a very limited region of space the intensity pattern is in antiphase to the main part. This is also visible in Fig. 2b) in the main text.

A. Band Structure calculation

As discussed in the main text, when computing the band structure of the total potential we assume that the deep axial lattice separates the full potential into a series of decoupled 2D problems. The resulting Schrödinger equation for a single atom in a 2D potential reads as

$$\left(-\frac{\hbar^2}{2m} \Delta + V(r, \phi) \right) \Psi(r, \phi) = E \Psi(r, \phi) \quad (6)$$

In the radial coordinates (energy is measured in units of the recoil energy $E_r = \frac{\hbar^2 k^2}{2m}$, $k = \frac{2\pi}{\lambda}$, the length unit is defined by setting $k = 1$) we obtain:

$$E\psi = -\left(\frac{1}{r} \partial_r + \partial_{rr} \right) - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + V(r, \phi) \psi. \quad (7)$$

We want to find the Bloch states ψ_q ,

$$\psi_q(r, \phi + \frac{\pi}{l}) = u_q(r, \phi) e^{i \frac{lq}{\pi} \phi}, \quad (8)$$

where u is a $\frac{\pi}{l}$ periodic function in the ϕ direction. Expanding ψ_q , in a Fourier series in the ϕ variable we get

$$u_q(r, \phi) = \frac{1}{\sqrt{2l}} \sum_{m=-\infty}^{\infty} e^{i2ml\phi} c_q(r, m) \quad (9)$$

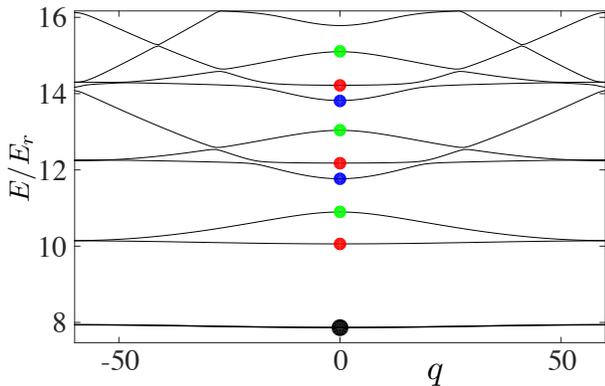


FIG. 6. (Color online) Bloch Band structure for a potential with ring geometry with ring geometry defined in terms of LG_{60,15} modes. Here $V_z = 20E_r, V_\phi = 5E_r$. The energy is measured relative to energy well depth. The almost-degenerate lowest excited Bloch bands are azimuthally excited (green), radially excited (red). Black dot marks the well-defined lowest Bloch band, while blue dots denote second azimuthally-excited bands. Soft radial modes spawn a whole ladder of states characterized by azimuthal excitation. Exact, formal identification of "radial" and "azimuthal" excitations is not possible, due to weak $r - \phi$ coupling.

where q is the quasi-momentum in azimuthal direction. The 2D Schrödinger equation (7) becomes then an eigenproblem expressed as a system of coupled differential equations in 1D real space. We solve these equations using sparse matrix diagonalization of a discretized real space equation.

The formal singularity of the eigenproblem (7) at $r = 0$ is resolved by imposing proper boundary conditions. Indeed, all components of Eq. (9) corresponding to a nonzero angular phase dependence must vanish at the origin, while the only component with no angular phase dependence, $c_0(r, 0)$ must satisfy: $\partial_r c_0(r, 0)|_{r=0} = 0$.

The numerical solution of the eigenproblem given by Eq. 6 gives rise to the Bloch spectrum shown in Fig. 6. It features a ground state separated from all excited states. As the problem Eq. 6 is weakly-nonseparable, the overall structure of the excited Bloch states in the spectrum is well described by elementary radial and azimuthal excitations, coupled by very small avoided crossings.

1. Tight-binding and Hubbard parameters

The purpose of this section is twofold. First, we microscopically compute the hopping integrals J_j^ϕ and J_j^z for the cylindrical optical lattice. Second, while the focus of the discussion in the main text is on spinless fermions, we here present the on-site Hubbard interaction parameters of the cylindrical lattice potential. These parameters are relevant if the cylindrical optical lattice is loaded with bosons which is experimentally equally conceivable.

From the numerical solution of Eq. 6 we can extract

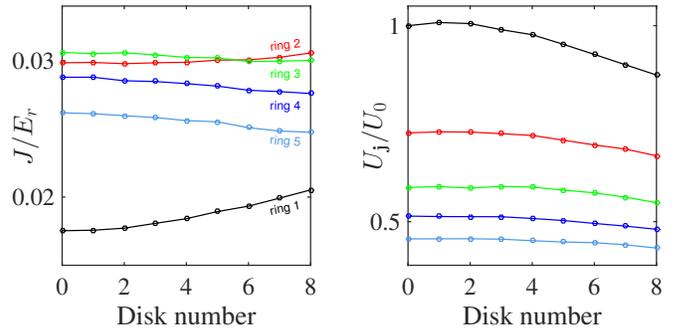


FIG. 7. (Color online) Panel a) shows the axial hopping amplitudes J_j^ϕ for different rings of the lattice potential. In this setup other hopping directions are prohibited by energy conservation. Panel b) shows the two-body interaction integrals U_j in the respective sites. The potential parameters are: $f/\omega_0 = 11$.

a set of Wannier functions in a standard way [2]. The Hubbard parameters describing the contact interaction $U_j = \frac{4\pi a \hbar^2}{m} \int d^3r |w_j(\vec{r})|^4$ and the hopping integrals J_j^ϕ and J_j^z are determined from band computation and construction of the Wannier functions.

The U_j, J_j parameters are constant on within each ring formed by lattice sites due to rotational symmetry.

In Fig. 7 we show curves denoting the U, J parameters for different potential rings as a function of disk number z to which the ring belongs to. Different curves show a data series for the innermost, second-innermost etc. rings. The radius of the lattice ring varies much weaker in the axial, z direction than in the radial direction, therefore the dependence of the curves on z is rather weak.

2. Rotation of the optical lattice

The flux Φ_z through the cylinder is implemented by rotation of the lattice in the azimuthal direction. A slight difference of frequencies of lasers creating the azimuthal optical lattice $\omega_1 - \omega_2 = \Omega$, causes its rotation around the optical axis with angular velocity Ω [1]. The time dependent transformation

$$U(t) = \exp\left(\frac{i}{\hbar} L_z \Omega t\right) = \exp\left(-\frac{\Omega t}{l} \frac{d}{d\phi}\right) \quad (10)$$

leads in the co-rotating frame to a modified Hamiltonian as defined by the Schrödinger equation

$$i\hbar \partial_t \psi = \left(H + i\Omega \hbar \frac{\partial}{\partial \phi}\right) \psi. \quad (11)$$

At the level of the tight-binding description, the extra term just amounts to a shift in the angular quasi-momentum. The corresponding shift of the Bloch bands is shown in Fig. 8a. Formally, this shift is reflected in a phase imprinted on the azimuthal hopping amplitudes

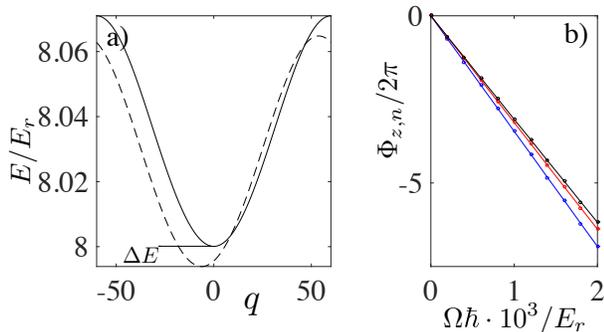


FIG. 8. (Color online) Panel a) shows effect of rotation on energy levels of band computation. Panel b) shows dependence of flux $\Phi_{z,n}$ threaded by imprinting the phase ϕ_z on azimuthal hoppings on the ring number n in cylinder configuration. Colors Black/Red/Blue indicate fluxes $\Phi_{z,0}$, $\Phi_{z,5}$, $\Phi_{z,10}$ as a function of Ω .

$J_z^\phi \rightarrow J_z^\phi e^{i\phi_{z,n}}$. The phases $\phi_{z,n}$ are slightly different for different potential disks. For the specific set of parameters presented in this work the dependence of $\phi_{z,n}$ on disk number n is presented in Fig 8. The rotation corresponds to threading an artificial magnetic field in z direction, and therefore the flux threaded through different rings depends on their radii.

For the simulation of the Laughlin experiment (see Fig. 3), we consider in this work a cylinder with an axial length of 11 sites around the focal plane. In this case, if the phase imprinted on a lattice ring in the focal plane is $\Phi_{z,0} = 10\pi$, the phase imprinted on the outermost rings (5th nearest neighbors) is $\Phi_{z,5} \approx 1.04\Phi_{z,0}$. This shows that the inhomogeneity of the probe flux is still negligible for ring of such a width.

B. Time of flight images

Time-of-flight (TOF) imaging is a standard experimental method to probe a single particle Green's function $G(r, r') = \langle \hat{\psi}(r, t) \hat{\psi}^\dagger(r', t) \rangle$ (see [3]). After the atomic cloud is abruptly released from the confinement and allowed to expand freely, an absorption imaging is performed along the optical axis z which measures the density of the gas after time t_{TOF} . The measured density is the full 3D density of the cloud, i.e., $n(x) = \left(\frac{m}{\hbar t_{TOF}}\right)^3 |\tilde{w}(\mathbf{k})|^2 \mathcal{G}(\mathbf{k} = \frac{m\mathbf{x}}{\hbar t_{TOF}})$, integrated along z direction, where w is the Wannier state of the occupied Bloch band.

Fig. 9 shows the absorption image for a single Bloch state calculated using the microscopic description of the potential modeled in this article (with $V_z = 20E_r$, $V_\phi = 5E_r$, $f/w_0 = 11$), and a fixed quasi-angular momentum $q \in \{-l+1, \dots, l\}$. It has the structure of a ring with a radius proportional to $|q|$. This allows to measure the distribution of the absolute value of the angular quasi-

momentum. Rotating the lattice as described in II A 2

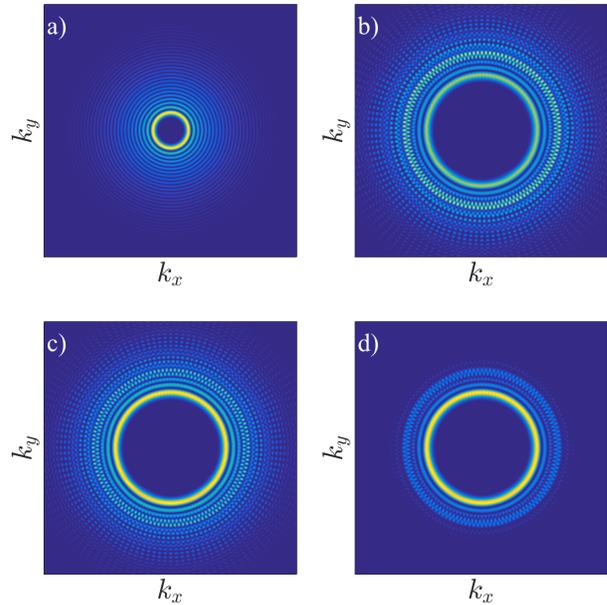


FIG. 9. (Color online) Simulated Time-of-flight images of a lowest band Bloch function in a cylindrical optical potential, released abruptly from the innermost ring of a potential in the focal plane. Here $V_z = 20E_r$, $V_\phi = 5E_r$, $f/w_0 = 11$. Panels show images of different quasi-angular momenta q . Subsequent panels show: a) $q = 15$, b) $q = 50$. Remaining panels show $q = 50$ as well in the case when the optical potential is switched off adiabatically, to allow for adiabatic band mapping [3]. Panel c) deals with case when adiabatic ramping down applies only to V_ϕ , while in panel d) both V_z, V_ϕ are decreased to final values $V_z = 0.5E_r$, $V_\phi = 0.5E_r$.

allows to shift the quasimomentum distribution and measure it as a function of $|q - q_0|$ with the experimentally controllable offset q_0 .

An alternative approach is to moderate the speed with which the lattice is switched off before the free expansion phase [3]. This enables the Bloch states to adiabatically follow the decreasing potential height. This converts the initial quasi-angular momentum for the deep lattice to the ordinary angular momentum distribution in the free space. This allows to reduce higher order peaks for large angular quasi-momenta q as seen in Fig. 9b).

In the presented model the potential is parametrized by two parameters: V_z and V_ϕ . We consider two cases of adiabatic ramping down of the potential. In the first case (see Fig. 9d), the initial azimuthal lattice depth $V_\phi = 5E_r$ is reduced to $V_\phi = 0.5$. In the second case (see Fig. 9d) the azimuthal lattice depth is decreased together with radial lattice depth to a final values of $V_\phi = 0.5$, $V_z = 0.5E_r$, which is then assumed to have been switched off abruptly (in the final stage when V_z is close to 0, adiabaticity is not possible).

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