Dissipative Chern Insulators

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Engineered dissipation can be employed to efficiently prepare interesting quantum many body states in a non-equilibrium fashion. Here, we study the open quantum system dynamics of fermions on a 2D lattice in the framework of a Lindblad master equation. In particular, we propose a novel mechanism to dissipatively prepare a topological state with non-zero Chern number as the unique steady state by means of *short-range* system bath interaction. This provides a genuine open quantum system approach to the preparation of topological states, which, quite remarkably, gives rise to a stable topological phase in a non-equilibrium phase diagram. We demonstrate how our theoretical construction can be implemented in a microscopic model that is experimentally feasible with cold atoms in optical lattices.

Introduction – The ability to engineer and control quantum many body systems in the laboratory provides us with possible novel scenarios for the realization of exotic quantum matter [1–3]. While most of the ongoing effort has focused on implementing many-body Hamiltonians in closed quantum systems, attention has recently turned to the design of open systems by engineering a desired system-bath coupling. This leads to the interesting question of *dissipatively* preparing exotic quantum states of matter [4–6]. A timely example is provided by topological states, which, in the Hamiltonian context of insulators and superconductors, have become a major focus of condensed matter physics [7–9]. Beyond attracting fundamental interest, topology entails a robustness towards imperfections which has been envisioned to be harnessed in technological applications. A driven-dissipative master equation dynamics to create topological states and Majorana edge modes has been given [10] for one-dimensional topological superconductors [11]. However, a generalization of this approach [10, 12] to higher dimensions, in particular to states with a non-vanishing Chern number [13–15], encounters the obstacle that *topology from dissipation* requires *non-local dissipation*. Below, we report a novel mechanism coined "dissipative hole-plugging" to overcome this issue. In particular, we demonstrate that *local* system bath engineering can result in a master equation with a unique steady state that is characterized by a non-vanishing Chern number for fermions on a 2D lattice. Quite remarkably, in our scheme, this topology from dissipation is robust, and exists as a stable topological phase in a non-equilibrium phase diagram in loose analogy to a gapped phase in the Hamiltonian context. Furthermore, we introduce a microscopic interacting model which in a mean field approximation reduces to the master equation studied in our general analysis. Such a scheme can be experimentally realized with cold atoms in optical lattices.

In our study we assume an open quantum system dynamics as described by a Lindblad master equation [16]

$$\dot{\rho} = i\left[\rho, H\right] + \sum_j \left(L_j \rho L_j^\dagger - \frac{1}{2}\left\{L_j^\dagger L_j, \rho\right\}\right) \qquad (1)$$

for the system density matrix ρ with the incoherent action of the Lindblad operators L_j accounting for the coupling of the system to a bath. We focus in the following on purely dissipative dynamics, i.e., we put $H = 0$ in Eq. (1). In analogy to the independent particle approximation for electrons in periodic potentials in the Hamiltonian context, we consider lattice translation invariant Lindblad operators L_j that are linear in the spinless [17] fermionic field operators $\psi_j, \psi_j^\dagger$ that form a complete fermionic algebra on a two dimensional (2D) lattice $\mathbb{Z}^2$. Also, we neglect initial many particle correlations by assuming a Gaussian initial state.

In the long time limit, the density matrix ρ approaches a stationary state described by a density matrix ρ^s. The generic recipe [10, 12] for preparing an arbitrary *pure* Gaussian steady state is the following. Construct a so called parent Hamiltonian $H_p = \sum_j L_j^\dagger L_j$ out of a complete set of anti-commuting Lindblad operators L_j. The ground state $|G\rangle\langle G|$ of this Hamiltonian will then be the unique steady state of the corresponding master equation (1). In other words, the Lindblad operators are chosen as single particle operators that span the many-body ground state of a quadratic Hamiltonian with the desired (topological) state as a unique ground state. In the Hamiltonian context, a topological ground state may be stabilized by a finite energy gap. The analog in the dissipative framework is a so called damping gap κ, which is given by the smallest rate at which deviations from the steady state ρ^s are damped out.

As we detail below, going beyond the realm of pure steady states turns out to be crucial here in order to stably prepare a steady state with non-vanishing Chern number by means of short-ranged Lindblad operators. For mixed steady states, topological properties over the Brillouin zone (BZ) associated with the lattice momentum are well defined as long as there is a finite purity gap [12], i.e., as long as the density matrix is not completely mixed at any lattice momentum. The topological steady states we are concerned with are protected by both a finite damping gap and a finite purity gap, i.e., their topology is unchanged under continuous deformations as long as both gaps are maintained.

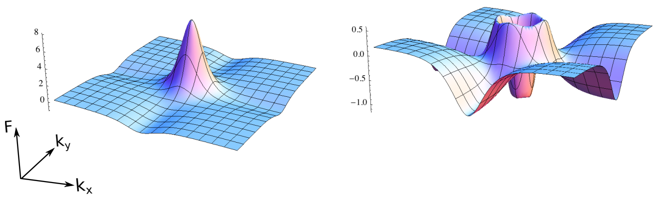

FIG. 1. (color online) Berry curvature F (integrand of Eq. (2)) as a function of momentum for $\beta = -3, d = g = 0$ (left) and $\beta = -3, d = 0.7, g = 2.0$ (right). The integral over the plotted function gives $\mathcal{C} = 0$ (left) and $\mathcal{C} = -1$ (right), respectively. In the left plot, the central peak compensates the smooth negative curvature away from the center. In the right plot the dissipative hole-plugging mechanism introduced here depletes the central peak to maintain the non-vanishing Chern number. Note the different scales in the plots.

The major conceptual challenge in the dissipative preparation of topological states in spatial dimension $d \geq 2$ is due to the competition of topology and locality of the operators L_j. To see this, consider a lattice translation invariant parent Hamiltonian that defines a (topological) band structure. An intuitive set of Lindblad operators L_j providing a real space representation of its many-body ground state then corresponds to the Wannier functions of this band structure. However, non-trivial topological invariants characterizing the ground state impose fundamental constraints on the localization properties of the Wannier functions [18–21] (see Ref. [22] for a detailed recent discussion). The archetype of a topological invariant for band structures is the first Chern number [13–15]

$$\mathcal{C} = \frac{i}{2\pi}\int_{\mathrm{BZ}} \mathrm{d}^2k \,\mathrm{Tr}\left(\mathcal{P}_k\left[(\partial_{k_x}\mathcal{P}_k),(\partial_{k_y}\mathcal{P}_k)\right]\right), \quad (2)$$

where $\mathcal{P}_k$ denotes the projection onto the occupied Bloch bands at lattice momentum k [23]. $\mathcal{C}$ is an integer quantized topological invariant of a gapped two-dimensional (2D) band structure. By its very definition, a non-vanishing Chern number implies an obstruction to finding a global smooth gauge for the associated family of Bloch functions [14, 15]. However, such a global smooth structure would be necessary to find an exponentially localized set of Wannier functions by Fourier transform of the Bloch functions. More precisely, it has been proven that exponentially localized Wannier functions exist if and only if the first Chern number is zero [18–20]. Hence, in order to dissipatively stabilize a topological state with a non-vanishing Chern number along the lines of Refs. [10, 12], i.e., by choosing a set of Lindblad operators that correspond to the Wannier functions of a parent Hamiltonian, long-ranged Lindblad operators with algebraic asymptotic decay properties would be inevitable. Here, going conceptually beyond the parent Hamiltonian analogy, we are able to overcome this issue and induce a steady state with non-vanishing Chern number by virtue of short-ranged Lindblad operators. To this end, we proceed in two steps. First, we construct an overcomplete set L^C of compactly supported Lindblad operators that yield a *critical* Chern state as a steady state. Second, we devise an auxiliary set of Lindblad operators L^A which is capable of lifting the topologically non-trivial critical point to an extended phase with a finite damping gap.

Correlation matrix for Gaussian states– Within the Gaussian approximation, all static information about the system state is contained in its Gaussian correlation matrix Γ which can be conveniently represented in the basis of the Majorana operators $c_{j,1} = \psi_j + \psi_j^\dagger,\ c_{j,2} = i(\psi_j^\dagger - \psi_j)$ as

$$\Gamma^{ij}_{\lambda\mu} = \frac{i}{2}\mathrm{Tr}\left\{\rho\left[c_{i,\lambda}, c_{j,\mu}\right]\right\}. \quad (3)$$

We focus on purely dissipative dynamics, i.e., we set $H = 0$ in Eq. (1). With the density matrix ρ evolving in time according to Eq. (1), Γ obeys the equation of motion [24, 25]

$$\dot{\Gamma} = \{\Gamma, X\} - Y, \quad (4)$$

where the matrices $X = M + M^T$ and $Y = 2i(M - M^T)$ are determined in terms of the Majorana representation of the Lindblad operators $L_j = \vec{l}_j^T\vec{c}$ via $M = \sum_j \vec{l}_j \vec{l}_j^\dagger$. Since we are dealing with lattice translation invariant systems here, it is convenient to consider the equation of motion for the Fourier transform of the correlation matrix, $\tilde{\Gamma}_{\lambda\mu}(k) = \frac{i}{2}\mathrm{Tr}\left\{\rho\left[c_{k,\lambda}, c_{-k,\mu}\right]\right\}$ which simply reads

$$\dot{\tilde{\Gamma}}(k) = \left\{\tilde{\Gamma}(k), \tilde{X}(k)\right\} - \tilde{Y}(k), \quad (5)$$

where $\tilde{X}, \tilde{Y}$ denote the Fourier transforms of X, Y. Eq. (5) is a 2×2 matrix equation for every lattice momentum k.

Over-completeness and damping-criticality – As a first step towards stabilizing a state with non-zero Chern number by virtue of short-ranged jump operators only, we resort to a set of non-orthonormal single particle operators coined pseudo Wannier functions as Lindblad operators instead of choosing proper Wannier functions. These pseudo Wannier functions still span a non-trivial many body state and have more benign localization properties. Concretely, let us consider the following Lindblad operators that correspond to a set of pseudo Wannier functions

$$L_j^C = \sum_i u^C_{i-j}\psi_i + v^C_{i-j}\psi_i^\dagger \quad (6)$$

with non-vanishing coefficients $v_0^C = \beta,\ v^C_{\pm\hat{x}} = v^C_{\pm\hat{y}} = 1$ and $u^C_{\pm\hat{x}} = -iu^C_{\pm\hat{y}} = \pm 1$, i.e., supported only on nearest neighbor sites. The associated (not normalised) Nambu pseudo Bloch functions $B_k^C \equiv (\tilde{u}_k^C, \tilde{v}_k^C)^T = (2i\left(\sin(k_x) + i\sin(k_y)\right), \beta + 2(\cos(k_x) + \cos(k_y)))^T$ are non-vanishing all over the BZ except at isolated values of β. At $\beta = -4$, for example, $(\tilde{u}_0^C, \tilde{v}_0^C) = (0, 0)$. However, the associated projection $\mathcal{P}_k = \frac{B_k^C B_k^{C\dagger}}{\mathrm{Tr}\left\{B_k^C B_k^{C\dagger}\right\}}$ approaches the value $P_0 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ as $k \to 0$ and can thus be defined continuously all over the BZ even for $\beta = -4$. Direct calculation of the associated Chern number using Eq. (2) yields $\mathcal{C} = -1$. However, $\mathcal{P}_k$ is not real analytic in $k = 0$ but only finitely differentiable. As a consequence $\mathcal{P}_k$ at $\beta = -4$ corresponds to a *critical* Chern-state that can have algebraically decaying correlations. Away from such isolated parameter-points, the pseudo Bloch functions are non-vanishing all over the BZ and the Chern number is zero. The criticality of the isolated topologically non-trivial points is also reflected in the closing of the damping gap $\kappa_k^C = |\tilde{u}_k^C|^2 + |\tilde{v}_k^C|^2 = \text{Tr}\left\{B_k^C B_k^{C\dagger}\right\}$. This phenomenology is very generic and not due to an unfortunate choice of our model as we will discuss now.

The pseudo Wannier functions corresponding to the Lindblad operators in Eq. (6) are in some analogy to the coherent states for the lowest Landau level (LLL) of electrons in a perpendicular magnetic field. These coherent states are overlapping exponentially localized pseudo Wannier functions that span the LLL. As has been extensively analyzed (see, e.g., Ref. [19]), they are necessarily overcomplete by one state due to the non-zero Chern number of the LLL. For the same reason, the above Bloch functions B_k^C must exhibit a zero associated with a critical damping behavior at the isolated topologically non-trivial points. Given its importance for our subsequent construction we briefly summarize the topological argument behind this observation. Consider any set of compactly supported or at least exponentially decaying orbitals $\phi_j(r)$ which are lattice translations of each other, i.e., $T_j\phi_0(r) = \phi_j(r) = \phi_0(r-j)$. Such pseudo Wannier functions need not be mutually orthogonal. Their lattice Fourier transforms $\tilde{\phi}_k$ are smooth functions that are not necessarily normalized. Still, as long as their norm is non-zero, they form different irreducible representations of the symmetry group of lattice translations and are hence mutually orthogonal. If the $\tilde{\phi}_k$ were non-vanishing for all k in the first Brillouin zone (BZ), one could continuously normalize them thus obtaining a global smooth gauge of Bloch functions – a contradiction to a non-vanishing Chern number as explained above. Hence, it is essential that $\|\tilde{\phi}_k\|$ has a zero which proves the over-completeness of any set of at least exponentially localized states spanning a Chern band. However, a dissipative preparation along these lines involves fine tuning since an infinitesimal deviation from a damping-critical topologically non-trivial point removes the essential zero from the associated Bloch functions, in turn rendering the state topologically trivial.

Dissipative hole-plugging mechanism – Deviating by δ from the topologically non-trivial critical point, i.e., $\delta = -4 - \beta$ may be viewed as tearing a hole into the smooth winding of $\mathcal{P}_k$ as a function of k (compare red dashed and green dotted left plots in Fig. 2): Indeed, a finite value of $\tilde{v}_0^C = \delta$ enforces $\mathcal{P}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ whereas at small $k > \delta$, $\mathcal{P}_k \approx \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. This rapid change in $\mathcal{P}_k$ compensates the almost complete smooth winding over the rest of the BZ (see Fig. 1) rendering the state trivial even at infinitesimal δ. To compensate for this "leak", we propose a dissipative hole-plugging mechanism which stabilizes the smooth winding of $\mathcal{P}_k$ even at finite deviations from the critical point. More specifically, we introduce the auxiliary Lindblad operators L_j^A which have only an annihilation part ($\tilde{v}_k^A = 0$) with, e.g., a Gaussian weight function $\tilde{u}_k^A = g e^{-k^2/d^2}$. These operators act selectively in momentum space: they prevent the unwanted occupation of the $k = 0$ mode as long as $g > \delta$, but their action becomes irrelevant for $k \gg d$. Alternative choices to the Gaussian weight function are conceivable, but this one makes makes the favorable localization properties manifest also in real space. In the presence of both sets of jump operators, L^C and L^A, we solve Eq. (5) for its steady state $\tilde{\Gamma}^s(k)$ and define the projection relevant for the calculation of the Chern number as [12]

$$\mathcal{P}_k = \frac{1}{2}(1 - i\bar{\Gamma}^s(k)), \tag{7}$$

where the hermitian traceless matrix $i\bar{\Gamma}^s(k) = \hat{n}(k)^i\tau_i$ results from $i\tilde{\Gamma}^s(k)$ by normalizing its eigenvalues to ± 1, $\hat{n}(k) \in S^2$ is a unit vector, and τ_i are Pauli-matrices in the Majorana pseudo spin space. This procedure is well defined as long as $\tilde{\Gamma}^s(k)$ has a finite purity gap, i.e., no zero eigenvalues [12]. Intuitively, $\bar{\Gamma}^s(k)$ is then the unique pure state that is closest to the steady state $\tilde{\Gamma}^s(k)$. We note that the Chern number associated with P_k (see Eq. 2) can also be calculated directly in terms of the 2×2 density matrix ρ_k without the deformation into a pure state, explicitly

$$\mathcal{C} = \frac{i}{2\pi}\int_{\text{BZ}} \frac{\text{Tr}\left\{\rho_k\left[(\partial_{k_x}\rho_k),(\partial_{k_y}\rho_k)\right]\right\}}{(2\text{Tr}\left\{\rho_k^2\right\}-1)^{\frac{3}{2}}}. \tag{8}$$

The denominator in Eq. (8) becomes singular if $\rho_k = \frac{1}{2}\mathbb{1}$ at some k which exactly corresponds to a purity gap closing where also $\mathcal{P}_k$ as in Eq. (7) is not well defined. In the following, we demonstrate that our hole plugging mechanism leads to a topologically non-trivial steady state in a finite parameter range around the critical point $\beta = -4$.

Benchmark results – We calculate the Chern number of the steady state obtained from the interplay of the Lindblad operators L_j^C and L_j^A while monitoring both its damping gap encoded in the eigenvalues of $\tilde{X}(k)$ and its purity gap given by the eigenvalues of $-\tilde{\Gamma}(k)^2$ [12]. We find that the additional auxiliary jump operators L_j^A are capable of lifting the isolated points at which the L_j^C become topologically non-trivial to an extended phase (see inset in right panel of Fig. 2). This hole plugging mechanism is illustrated in terms of the Berry curvature $F = \text{Tr}\left(\mathcal{P}_k\left[(\partial_{k_x}\mathcal{P}_k),(\partial_{k_y}\mathcal{P}_k)\right]\right)$, i.e., the integrand of Eq. (2) in Fig. 1: The central peak in the Berry curvature destroying the non-trivial Chern number is suppressed by the action of the L_j^A jump operators thus maintaining $\mathcal{C} = -1$. If we start in the absence of L_j^A with a topologically trivial $\delta = -4 - \beta$, i.e., detuned from the critical point by $\delta > 0$ and switch on L_j^A by ramping up g, we

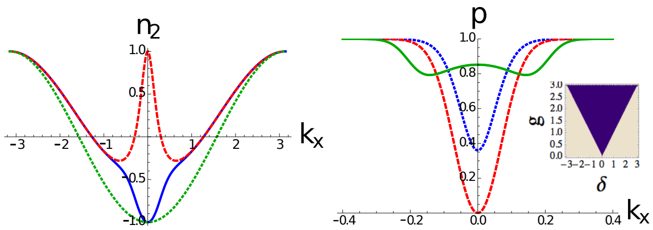

FIG. 2. (color online) Left panel: $\hat{n}^2 = -\mathrm{Tr}\left\{i\tilde{\Gamma}^s\tau_2\right\}$ (see Eq. (7)) as a function of k_x at $k_y = 0$ for $\delta = 0$ (green dotted), for $\delta = 0.5, d = g = 0$ (Red dashed), and for $\delta = 0.5, d = 0.7, g = 1.0$ (blue solid). Right panel: Purity p of the steady state $\tilde{\Gamma}^s(k)$ as a function of k_x at $k_y = 0$. $\delta = d = 0.2$ in all plots. Gap at $g = 0.1$ (blue dotted) in the topologically trivial phase, critical point at $g = 0.2$ (red dashed), gap at $g = 1.0$ (green solid) in the topologically non-trivial phase. Inset: Phase diagram of the steady state as a function of $\delta = -4 - \beta$ and g. $d = 1.0$ is fixed. The purple region has Chern number $\mathcal{C} = -1$, while $\mathcal{C} = 0$ in the bright region. The purity gap closes at the transition lines. The damping gap is finite everywhere except at the critical point $\delta = g = 0$.

observe a topological transition associated with a purity gap closing at $g = \delta$ (see Fig. 2). At $g > \delta$, the purity gap reopens and the steady state has Chern number $\mathcal{C} = -1$. The damping gap stays finite throughout this procedure. If we start right from the critical point $\beta = -4$ and switch on g, the damping gap opens continuously and the purity gap never closes. In this case $\mathcal{C} = -1$ throughout the process. If then, at finite g, a $\delta < g$ is switched on continuously, no topological phase transition is observed and the Chern number stays unchanged at $\mathcal{C} = -1$.

Microscopic implementation – We now introduce a microscopic, particle number conserving model described by a master equation that is quartic in the field operators, and we argue how this model is experimentally feasible with cold atoms in optical lattices. As we confirm numerically, the phenomenology described above is obtained in a mean field approximation analogous to the one introduced in Ref. [10]. The model consists of a near-critical and -topological set of Lindblad operators ℓ^C, and an auxiliary set ℓ^A which achieves the hole plugging discussed above. The Lindblad operators have the number conserving bilinear form $\ell_i^\alpha = C_i^{\alpha\,\dagger} A_i^\alpha$, $\alpha = C, A$ with the creation $C_i^{\alpha\,\dagger} = \sum_j v_{j-i}^\alpha \psi_j^\dagger$ and annihilation parts $A_i^{\alpha\,\dagger} = \sum_j u_{j-i}^\alpha \psi_j$. For $\alpha = C$, the coefficients u_j^C, v_j^C are the same as in Eq. (6). The experimental implementation of Lindblad operators of such a form has been discussed in Ref. [12]. The auxiliary operators ℓ_j^A are chosen such that particles are pumped out of the central region of the Brillouin zone into the higher momentum states thus reflecting the depletion of low momenta which is at the heart of our hole plugging mechanism. For atoms in optical lattices this can be achieved by momentum selective pumping techniques as described in [26]. In momentum space, $\tilde{\ell}_k^A = \sum_q \tilde{C}_{q-k}^{A\,\dagger} \tilde{A}_q^A$, with $\tilde{C}_k^\dagger = \tilde{v}_k^A \tilde{\psi}_k^\dagger, \tilde{A}_k^A = \tilde{u}_k^A \tilde{\psi}_k$. The momentum selective functions are ideally of the form $\tilde{u}_k^A = g_u e^{-k^2/d_u^2}$ removing particles from the central region, and $\tilde{v}_k^A = g_v \sum_i e^{-(k-\pi_i)^2/d_v^2}$ describing their reappearance at high momenta $\pi_i \in \{(0,\pi), (\pi,0), (\pi,\pi)\}$. The key qualitative point to the form of ℓ_k^A that has to be reflected in an approximate experimental realization is the dominance of processes taking a particle at the center of the Brillouin zone and transferring a momentum of order π.

In a mean field decoupling, the product of the creation and annhilation part in ℓ_i^C can be linearized and transforms into its sum [10], yielding precisely the form displayed in Eq. (6) at half filling. This allows us to evaluate the stationary state of the master equation with both sets of Lindblad operators ℓ^C, ℓ^A at mean field level (see Supplementary Material for technical details). The above general picture, in particular the efficiency of the hole-plugging mechanism to stabilize a state with non-vanishing Chern number is fully confirmed by the numerical analysis of this microscopic model.

Conclusion – We have proposed a mechanism to dissipatively stabilize topological states of quantum matter with non-vanishing Chern number. Our generally applicable construction relies on short ranged system bath interaction only. An initial set of strictly local Lindblad operators L^C can be fine-tuned to yield a critical Chern state. By virtue of an auxiliary set of short ranged Lindblad operators L^A, a gapped state with non-vanishing Chern number is stabilized in an extended parameter range, i.e. without fine-tuning. However, the resulting state is in general mixed with a finite purity gap but can be brought arbitrarily close to a pure state by adjusting the model parameters. Relying on the combination of these two sets of jump operators, this scheme crucially exploits the open quantum system character of the problem and hence goes conceptually beyond the Hamiltonian analog. The target state of the explicit construction presented here resembles the $p + ip$ superconducting ground state introduced by Read and Green [27], i.e., a topological state with non-vanishing Chern number in symmetry class D [28]. The generalization to gapped quantum anomalous Hall states aka Chern insulators [29] in symmetry class A, however, is straightforward.

The criticality of the steady state supported by the L^C operators is in some analogy to Ref. [30], where a tensor network state representing a critical Chern state is constructed. While it may be impossible to represent pure non-critical states with non-vanishing Chern number as tensor network states, resorting to mixed states that are uniquely associated with a pure state via Eq. (7) might be fruitful for systematic approximations of such scenarios, where the deviation from the gapped pure state in the expectation value of any observable can be bounded in terms of the purity gap.

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SUPPLEMENTARY MATERIAL: SELF-CONSISTENT MEAN FIELD THEORY

A self-consistent mean field theory can be derived along the lines of [12]. We start from the master equation

$$\partial_t \rho = \mathcal{L}^C[\rho] + \mathcal{L}^A[\rho], \tag{9}$$

where $\mathcal{L}^\alpha[\rho] = \sum_j \left(\ell_j^\alpha \rho \ell_j^{\alpha\,\dagger} - \frac{1}{2}\left\{\ell_j^{\alpha\,\dagger}\ell_j^\alpha, \rho\right\}\right), \quad \alpha = C, A$. We write out this quartic master equation in momentum space, and make the ansatz $\rho = \prod_k' \rho_k$, where ρ_k describes the mode pair $\{k, -k\}$ and obeys $\mathrm{Tr}_k \rho_k = 1$. $\prod_k'$ reminds that the product is taken over half of the Brillouin zone only, e.g. the upper half. We then focus on one particular momentum mode pair $\{p, -p\}$, and keep only terms which are quadratic in operators associated to this mode pair. By means of the prescription $\rho_p = \mathrm{Tr}_{\neq p}\mathcal{L}[\rho]$, we obtain an evolution equation for ρ_p with coefficients C_i^α, which are governed by the mean fields of the remaining modes in the system,

$$\begin{aligned}\partial_t \rho_p = \sum_{\alpha=C,A} \Big\{ & C_1^\alpha |\tilde{v}_p^\alpha|^2 \mathcal{L}_{a_p^\dagger, a_p}[\rho_p] + C_2^\alpha |\tilde{u}_p^\alpha|^2 \mathcal{L}_{a_p, a_p^\dagger}[\rho_p] \\ & - C_3^\alpha \tilde{v}_p^\alpha \tilde{u}_{-p}^{\alpha\,*} \mathcal{L}_{a_p^\dagger, a_{-p}^\dagger}[\rho_p] - C_3^{\alpha\,*} \tilde{u}_{-p}^\alpha \tilde{v}_p^{\alpha\,*} \mathcal{L}_{a_{-p}, a_p}[\rho_p] \\ & + \{p \to -p\} \Big\},\end{aligned} \tag{10}$$

with abbreviation $\mathcal{L}_{a,b}[\rho] = a\rho b - \frac{1}{2}\{ba, \rho\}$ and

$$\begin{aligned} C_1^\alpha &= \sum_{q\neq p} |\tilde{u}_q^\alpha|^2 \langle a_q^\dagger a_q\rangle, \; C_2^\alpha = \sum_{q\neq p} |\tilde{v}_q^\alpha|^2 (1 - \langle a_q^\dagger a_q\rangle), \\ C_3^\alpha &= \sum_{q\neq p} \tilde{u}_q^\alpha \tilde{v}_{-q}^{\alpha\,*} \langle a_{-q} a_q\rangle; \end{aligned} \tag{11}$$

we note $\langle a_{-q} a_q\rangle^* = \langle a_q^\dagger a_{-q}^\dagger\rangle$, and that the constraint on the sum can be neglected in the thermodynamic limit.

In the absence of $\mathcal{L}^A$, the stationary state is known explicity, using the equivalence of fixed number and fixed phase wavefunctions in the thermodynamic limit, and the exact knowledge of the fixed number state annihilated by the set ℓ_i^C [12]. It is given by the pure density matrix $\rho_D = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \prod_q' \mathcal{N}_q (1 + \frac{\tilde{v}_q^C}{\tilde{u}_q^C} a_{-q}^\dagger a_q^\dagger)|0\rangle$ and $\mathcal{N}_q = 1/\sqrt{1 + |\tilde{v}_q^C/\tilde{u}_q^C|^2}$. In this case, the averages can be evaluated explicitly, $\langle a_q^\dagger a_q\rangle = |\bar{v}_q^C|^2, \langle a_q a_q^\dagger\rangle = |\bar{u}_q^C|^2, \langle a_{-q} a_q\rangle = \bar{v}_{-q}^C \bar{u}_q^{C\,*}, \quad \langle a_q^\dagger a_{-q}^\dagger\rangle = \bar{v}_{-q}^{C\,*} \bar{u}_q^C$, where $\bar{v}_q^C = \tilde{v}_q^C/\sqrt{|\tilde{u}_q^C|^2 + |\tilde{v}_q^C|^2}$ and analogous for $\bar{u}_q^C$. Note that, in this case, one common real number $C_0 = C_1^C = C_2^C = C_3^C = C_3^{C\,*} = \sum_{q\neq p} \frac{|\tilde{u}_q^C \tilde{v}_q^C|^2}{|\tilde{u}_q^C|^2 + |\tilde{v}_q^C|^2}$ can be factored out of Eq. (10), and the resulting linearized Lindblad operators coincide with those defined in Eq. (6) in the main text (at half filling, and up to an irrelevant relative phase which reflects spontaneous symmetry breaking).

When $\mathcal{L}^A$ is added, the stationary state is no longer known explicitly. However, a self-consistent mean field theory can be constructed. To this end, we derive the evolution of the covariances for the mode pair $\{p, -p\}$,

$$\partial_t \begin{pmatrix} \langle a_p^\dagger a_p\rangle \\ \langle a_{-p} a_p\rangle \\ \langle a_p^\dagger a_{-p}^\dagger\rangle \end{pmatrix} = \begin{pmatrix} -\kappa_p & \nu_p^* & \nu_p \\ 0 & -\kappa_p & 0 \\ 0 & 0 & -\kappa_p \end{pmatrix} \begin{pmatrix} \langle a_p^\dagger a_p\rangle \\ \langle a_{-p} a_p\rangle \\ \langle a_p^\dagger a_{-p}^\dagger\rangle \end{pmatrix} + \begin{pmatrix} \mu_p \\ \lambda_p \\ \lambda_p^* \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} \kappa_p &= \sum_\alpha (C_1^\alpha |\tilde{v}_p^\alpha|^2 + C_2^\alpha |\tilde{u}_p^\alpha|^2), \\ \nu_p &= \tfrac{1}{2} \sum_\alpha C_3^\alpha (\tilde{v}_{-p}^\alpha \tilde{u}_p^{\alpha\,*} + \tilde{v}_p^\alpha \tilde{u}_{-p}^{\alpha\,*}), \\ \mu_p &= \sum_\alpha C_1^\alpha |\tilde{v}_p^\alpha|^2, \; \lambda_p = \tfrac{1}{2}\sum_\alpha C_3^\alpha (\tilde{v}_{-p}^\alpha \tilde{u}_p^{\alpha\,*} - \tilde{v}_p^\alpha \tilde{u}_{-p}^{\alpha\,*}). \end{aligned} \tag{13}$$

Here we have used the property $|\tilde{v}_p^\alpha|^2 = |\tilde{v}_{-p}^\alpha|^2$ and analogous for $\tilde{u}_p$, exhibited by our Lindblad operators. The coefficients C_i^α make these equations non-linear. The implicit equations for the stationary state read as

$$\begin{pmatrix} \langle a_p^\dagger a_p\rangle \\ \langle a_{-p} a_p\rangle \\ \langle a_p^\dagger a_{-p}^\dagger\rangle \end{pmatrix} = \frac{1}{\kappa_p} \begin{pmatrix} 1 & \frac{\nu_p^*}{\kappa_p} & \frac{\nu_p}{\kappa_p} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_p \\ \lambda_p \\ \lambda_p^* \end{pmatrix}. \tag{14}$$

These equations can be solved iteratively, starting from the known solution of $\mathcal{L}^C$ alone. Qualitative properties of the solution can be discussed on the basis of the localization properties of the functions $\tilde{u}_q^A, \tilde{v}_q^A$ in momentum space. In particular, based on Eq. (10), we expect modifications of the solution for $\mathcal{L}^C$ only in the central and edge regions of the Brillouin zone, as clearly effective annhilation (creation) of particles takes place in the center (edges) of the Brillouin zone. In addition, the off-diagonal contributions to the effective non-topological Liouvillian are exponentially small.

In Fig. 3, we give an explicit example demonstrating how the self consistent solution of Eq. (14) is capable of achieving

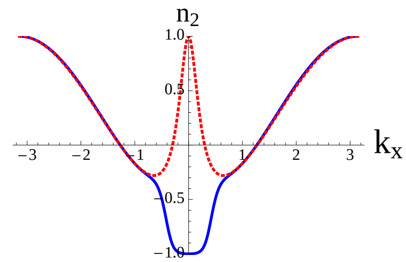

FIG. 3. $\hat{n}^2 = -\mathrm{Tr}\left\{i\bar{\Gamma}^s \tau_2\right\}$ (see Eq. (7)) as a function of k_x at $k_y = 0$. Red dashed plot for $\delta = -4 - \beta = 0.5$ in the presence of ℓ_j^C only. Blue solid plot shows the self-consistent solution of Eq. (14) for $\delta = 0.5, g_u = g_v = 5.0, d_u = 0.5, d_v = 1.5$ in the presence of both ℓ_j^C and ℓ_j^A on a lattice of 501×501 sites.

the hole plugging mechanism. Our numerical simulations are done for a lattice of 501×501 sites with periodic boundary conditions.

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