Two-dimensional superfluidity in driven systems requires strong anisotropy

Ehud Altman
Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel

John Toner
Department of Physics and Institute of Theoretical Science, University of Oregon, Eugene OR, 97403, U.S.A.

Lukas M. Sieberer and Sebastian Diehl
Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria and Institute for Quantum Optics and Quantum Information of the Austrian Academy of Sciences, A-6020 Innsbruck, Austria

Leiming Chen
College of Science, The China University of Mining and Technology, Xuzhou Jiangsu, 221116, P.R. China

We show that driven two-dimensional Bose systems cannot exhibit algebraic superfluid order unless the underlying microscopic system is strongly anisotropic. Our result implies, in particular, that recent apparent evidence for Bose condensation of exciton-polaritons in semiconductor quantum wells must be an intermediate scale crossover phenomenon, while the true long distance correlations fall off exponentially. We obtain these results through a mapping of the long-wavelength condensate dynamics onto the anisotropic Kardar-Parisi-Zhang equation.

One of the most striking discoveries to emerge from the study of non-equilibrium systems is that they sometimes exhibit ordered states that are impossible in their equilibrium counterparts. For example, it has been shown [1] that a two-dimensional "flock" - that is, a collection of moving, self-propelled entities - can develop long-ranged orientational order in the presence of finite noise (the non-equilibrium analog of temperature), and in the absence of both rotational symmetry breaking fields and long ranged interactions. In contrast, a two-dimensional *equilibrium* system with short-ranged interactions (e.g., a two-dimensional ferromagnet) cannot order at finite temperature; this is the Mermin-Wagner theorem [2].

In this paper, we report an example of the opposite phenomenon: A *driven*, two-dimensional Bose system, such as a gas of polariton excitations in a two-dimensional isotropic quantum well [3], cannot exhibit off-diagonal algebraic correlations (i.e., two-dimensional superfluidity). In the polariton gas, the departure from thermal equilibrium is due to the incoherent pumping needed to counteract the intrinsic losses and maintain a constant excitation density.

The critical properties of related driven quantum systems have been the subject of numerous theoretical studies; in certain cases it can be shown that the low frequency correlation functions induced by driving are identical to those in equilibrium systems at an effective temperature set by the driving [4–8]. Such emergent thermalization occurs in three dimensional bosonic systems, although non-equilibrium effects *can* change the *dynamical* critical behavior [8]. Here, we show that the non-equilibrium conditions imposed by the driving have a much more dramatic effect on *two-dimensional* Bose systems: effective equilibrium is never established in the generic isotropic case; instead, the non-equilibrium nature of the fluctuations inevitably destroys the condensate at long scales.

Our results suggest that recent experiments [9–12] with semiconductor quantum wells purporting to show evidence for the long sought [13] Bose condensation of polariton excitations are in fact observing an intermediate length scale crossover phenomenon, and not the true long-distance behavior of correlations. This conclusion rests on an exact mapping, which we derive below, between the long wavelength dynamics of a driven condensate and the Kardar-Parisi-Zhang (KPZ) equation [14] originally formulated to describe randomly growing interfaces. The non-equilibrium fluctuations generated by the drive translate into the non-linear terms of the KPZ equation.

Using the same mapping we show that effective equilibrium *can* be established *only* if the four- or six-fold rotational symmetry is strongly broken; this does not, to our knowledge, happen in any of the semiconductor quantum wells used in current experiments. In this case, algebraic order can exist at long scales, as in an equilibrium superfluid at finite temperature, and the transition to the disordered phase occurs by a standard equilibrium Kosterlitz-Thouless transition.

Our results also apply to a far broader class of systems; namely, any system described by the noisy complex Ginzburg-Landau equation [15, 16]. This includes, in addition to the driven BEC problem that motivated us, nonlinear chemical waves [17], driven superconductors [4], driven noisy oscillators modelling the physics of hearing [18, 19], and a variety of pattern-forming dynamical systems [20].

Model – The simplest model for Bose-Einstein condensation has a complex scalar order parameter field ψ, which can be thought of as the macroscopic wavefunction of the condensate. Since we are interested in a driven, non-equilibrium system, we cannot proceed as in equilibrium, i.e., by considering the partition function associated with a "Landau-Ginzburg-Wilson free energy" that is a functional of ψ. Rather, we must formulate an equation of motion consistent with the symmetries of the systems. Here, the symmetries are translation invariance and the usual invariance under uniform changes in the phase of the wavefunction: $\psi \to \psi e^{i\phi}$, where ϕ is an arbitrary constant. In order for the latter to remain a symmetry for short-lived excitations such as polaritons, the driving field which maintains them at a constant density must be incoherent, so that it does not imprint a phase on the polaritons it excites.

The dynamics of a driven-dissipative system like a polariton condensate is determined both by coherent processes, such as the dispersion and scattering between polaritons, and independent dissipative processes induced by loss and pump fields. A model of the condensate dynamics that incorporates these processes and respects the aforementioned symmetries is [8]:

$$\partial_t \psi(\mathbf{x}, t) = -\frac{\delta H_d}{\delta \psi^*} + i\frac{\delta H_c}{\delta \psi^*} + \zeta(\mathbf{x}, t). \tag{1}$$

Here

$$H_\ell = \int dx\ dy \left[r_\ell|\psi|^2 + K_\ell^x|\partial_x\psi|^2 + K_\ell^y|\partial_y\psi|^2 + \frac{1}{2}u_\ell|\psi|^4 \right] , \tag{2}$$

with $\ell = c, d$ are the effective Hamiltonians that generate the coherent and dissipative dynamics respectively. The last term $\zeta(\mathbf{x}, t)$ in Eq. (1) is a zero mean Gaussian white noise with short-ranged spatiotemporal correlations: $\langle\zeta^*(\mathbf{x}, t)\zeta(\mathbf{x}', t')\rangle = 2\sigma\delta^d(\mathbf{x} - \mathbf{x}')\delta(t - t')$, $\langle\zeta(\mathbf{x}, t)\zeta(\mathbf{x}', t')\rangle = 0$.

Eq. (1) is widely known as the complex Ginzburg-Landau equation [16, 20], or in the context of polariton condensates, as the dissipative Gross-Pitaevskii equation [21, 22], although usually only the isotropic (i.e., $K_\ell^x = K_\ell^y$), noise free ($\zeta = 0$) case is considered (but see [15]). Modifications of this equation, e.g., including higher powers of ψ and ζ, higher derivatives, or combinations of the two, can readily be shown to be irrelevant in the Renormalization Group (RG) sense. That is, they have no effect on the long-distance, long-time scaling properties of either the ordered phase, or the transition into it [23].

Each of the parameters appearing in the model has a clear physical origin, as we now review. The coefficient r_d is the single particle loss rate γ_l (spontaneous decay) offset by the pump rate γ_p, that is, $r_d = \gamma_l - \gamma_p$. In contrast, r_c is an effective chemical potential, which is *completely arbitrary*. By changing variables $\psi(\mathbf{x}, t) = \psi'(\mathbf{x}, t)e^{i\omega t}$ we can obtain Eq. (1) for $\psi'(\mathbf{x}, t)$, with only r_c changed, by $r_c \to r_c - \omega$. Thus, by a suitable choice of ω, we can make r_c take on any value we like. This is very similar to a "gauge choice". We will henceforth choose r_c' so that, in the absence of noise, the equation of motion has a stationary, spatially uniform solution. We will also drop the prime in the following.

The term proportional to u_c is the pseudo-potential which describes the elastic scattering of two polaritons, whereas u_d is the non-linear loss that ensures saturation of particle number. The coefficients $K_c^{x,y} = \hbar^2/(2m_{x,y})$, where $m_{x,y}$ are the eigenvalues of the effective polariton mass tensor, with principal axes x, y. Under typical circumstances, the diffusion-like term K_d is expected to be small, but it is allowed by symmetry, and so will always be generated [24, 25]. Finally, the noise is given by the total rate of particles entering and leaving the system: $2\sigma \approx \gamma_p + \gamma_l + u_d\bar{n}$, where $\bar{n}$ is the average density. In mean field theory, $\bar{n} = |\langle\psi(\mathbf{x}, t)\rangle|^2$ [26].

Before proceeding, it is important to clarify under what conditions Eq. (1) describes an effective thermal equilibrium at *all* wavelengths. If the steady state of such a stochastic equation is to be described by a Gibbs distribution of the field, it should satisfy a specific relation between the Hamiltonian component of the evolution generated by Poisson-brackets and the dissipative component [15, 27–29]. Applied to the problem at hand, with Poisson brackets $\{\psi(\mathbf{x}), \psi(\mathbf{x}')^*\} = \delta(\mathbf{x} - \mathbf{x}')$, the equilibrium condition translates to the simple requirement $H_d = RH_c$, where R is a multiplicative constant.

In contrast, in a driven system, the dissipative and coherent parts of the dynamics are generated by independent processes. Hence, in a driven system this relation $H_d = RH_c$ will not hold in general, although it can arise as an emergent symmetry at long times and wavelengths. This was shown to be the case for a three-dimensional driven condensate [8]. Below we shall derive the hydrodynamic long-wavelength description of a two-dimensional driven condensate and determine if it flows to effective equilibrium.

Mapping to a KPZ equation – We will now show that the model (1) can be mapped, in the long-wavelength limit, onto a KPZ equation [14]. As in equilibrium, the first step in deriving a hydrodynamic description of the condensate is to write the order-parameter field in the amplitude-phase representation as $\psi(\mathbf{x},t) = (M_0 + \chi(\mathbf{x},t))e^{i\theta(\mathbf{x},t)}$, with M_0, χ, and θ all real. Here M_0 is determined by requiring that $\chi = 0$, $\theta = 0$ is a static uniform solution of Eq. (1) in absence of fluctuations ($\zeta(\mathbf{x},t) = 0$). The real and imaginary parts of Eq. (1) then give $M_0^2 = -r_d/u_d$ and $r_c = -u_c M_0^2$ respectively. We can satisfy the second condition by exploiting our aforementioned freedom to chose r_c. As explained above, by varying the strength of the pump laser, one can experimentally control r_d, which determines the amplitude M_0. The mean field transition occurs at the point $r_d = 0$ (i.e., when $\gamma_p = \gamma_l$), where the amplitude M_0 vanishes. For later convenience we define the dimensionless tuning parameter $x \equiv \gamma_p/\gamma_l - 1$.

Plugging the amplitude-phase representation of ψ into Eq. (1), and linearizing in the amplitude fluctuations χ, we obtain the pair of equations

$$\partial_t \chi = -2u_d M_0^2 \chi - K_c^x M_0 \partial_x^2 \theta - K_c^y M_0 \partial_y^2 \theta - K_d^x M_0 (\partial_x \theta)^2 - K_d^y M_0 (\partial_y \theta)^2 + \mathrm{Re}\zeta \ , \tag{3}$$

$$M_0 \partial_t \theta = -2u_c M_0^2 \chi + K_d^x M_0 \partial_x^2 \theta + K_d^y M_0 \partial_y^2 \theta - K_c^x M_0 (\partial_x \theta)^2 - K_c^y M_0 (\partial_y \theta)^2 + \mathrm{Im}\zeta \ , \tag{4}$$

where we have used the freedom discussed earlier to choose $r_c = -u_c M_0^2$ to simplify this expression.

Note that if we have no dissipation ($H_d = 0$), so that $u_d = 0$, both χ and θ are "slow" variables, in the sense of evolving at rates that vanish as the wavevector goes to zero. In this case we can substitute Eq. (3) into the time derivative of Eq. (4) to obtain a wave equation for θ supplemented by irrelevant non-linear corrections. This gives the linear dispersion of the undamped Goldstone modes characteristic of a lossless condensate with exact particle number conservation. In contrast, without particle number conservation (i.e., in the presence of loss and drive), $u_d \neq 0$, and we can therefore neglect the $\partial_t \chi$ term (which vanishes as frequency $\omega \to 0$) on the left hand side of Eq. (3) relative to the $2u_d M_0^2 \chi$ on the right hand side for any "hydrodynamic mode" (i.e., in the low frequency limit). Doing so turns Eq. (3) into a simple linear algebraic equation relating χ to spatial derivatives of θ. Substituting the solution for χ of this equation into Eq. (4) gives a closed equation for θ

$$\partial_t \theta = D_x \partial_x^2 \theta + D_y \partial_y^2 \theta + \frac{\lambda_x}{2}(\partial_x \theta)^2 + \frac{\lambda_y}{2}(\partial_y \theta)^2 + \bar{\zeta}(\mathbf{x},t) \ , \tag{5}$$

with ($\alpha = x, y$):

$$D_\alpha = K_d^\alpha \left[1 + \frac{K_c^\alpha}{K_d^\alpha}\frac{u_c}{u_d}\right] \ , \tag{6}$$

$$\lambda_\alpha = 2K_c^\alpha \left[\frac{K_d^\alpha}{K_c^\alpha}\frac{u_c}{u_d} - 1\right] \ .$$

Here we have included all terms that are marginal and relevant by canonical power-counting, while neglecting irrelevant terms like $\partial_t^2\theta$, $\partial_t \nabla^2 \theta$, and $\partial_t (\nabla\theta)^2$. The noise in this equation is related to the original noise through $\bar{\zeta} = (\mathrm{Im}\zeta - u_c \mathrm{Re}\zeta/u_d)/M_0$. Hence $\langle \bar{\zeta}(\mathbf{x},t)\bar{\zeta}(\mathbf{x}',t')\rangle = 2\Delta \delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t')$ with

$$\Delta = \frac{\sigma(1 + u_c^2/u_d^2)}{2M_0^2} = (1 + u_c^2/u_d^2)\frac{\gamma_p + \gamma_l + u_d M_0^2}{4M_0^2} = \frac{(u_d^2 + u_c^2)\gamma_p}{2u_d(\gamma_p - \gamma_l)} \ . \tag{7}$$

Eq. (5) is the anisotropic KPZ equation, originally formulated to describe the roughness of a growing surface due to random deposition of particles on it [14, 30], in which case θ is the height of the interface. It reduces to the *isotropic* KPZ equation when $D_x = D_y$ and $\lambda_x = \lambda_y$. This reduction can also be achieved by a trivial rescaling of lengths if $\Gamma \equiv \frac{\lambda_y D_x}{\lambda_x D_y} = 1$. Thus, when $\Gamma \neq 1$, the system is anisotropic.

It is important to note that our KPZ model differs from that formulated for a description of randomly growing interfaces [14] in that the analog of the interface height variable in our model is actually a compact phase; hence, topological defects in this field are possible. This difference with the conventional KPZ equation also arises in "Active Smectics" [31].

Analysis of Eq. (5) in absence of vortices is the analogue of the low temperature spin-wave (linear phase fluctuation) theory of the equilibrium XY model. The crucial difference here is the appearance of the nonlinear terms with coefficients λ_x, λ_y, which are a direct measure of the deviation from thermal equilibrium. They vanish identically when the equilibrium conditions

$$K_c^x/K_d^x = K_c^y/K_d^y = u_c/u_d \ , \tag{8}$$

which follow from the equilibrium requirement that $H_d = RH_c$, are met. Indeed, without the non-linear terms, the KPZ equation reduces to linear diffusion, which would bring the field to an effective thermal equilibrium with power-law off-diagonal correlations (in $d = 2$). A transition to the disordered phase in this equilibrium situation can occur only as a Kosterlitz-Thouless (KT) transition through proliferation of topological defects in the phase field.

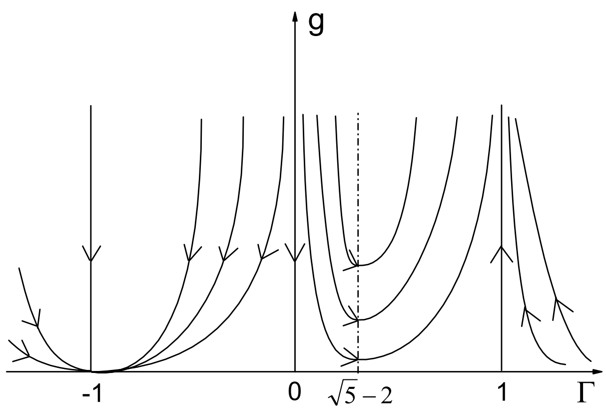

FIG. 1. The RG flow in the Γ-g parameter space for anisotropic driven BEC in $d = 2$. For $\Gamma < 0$ and $g > 0$, all flow lines go to a stable fixed point $(-1, 0)$; for $\Gamma > 0$ and $g > 0$, all flow lines go to infinity, and approach the isotropic limit $\Gamma = 1$.

In a driven condensate, the non-linear terms are in general present, and in two dimensions have the same canonical scaling dimension as the linear terms. A more careful RG analysis is therefore required to determine how the system behaves at long scales even without defect proliferation. Such an analysis has been done in Refs. [30] and [31] for the anisotropic KPZ equation. In this case, the flow is closed in the two parameter space of scaled non-linearity $g \equiv \frac{\lambda_x^2 \Delta}{D_x^2 \sqrt{D_x D_y}}$ and scaled anisotropy $\Gamma \equiv \frac{\lambda_y D_x}{\lambda_x D_y}$, and is given, to leading order in g, by:

$$\frac{dg}{dl} = \frac{g^2}{32\pi}(\Gamma^2 + 4\Gamma - 1),$$
$$\frac{d\Gamma}{dl} = \frac{\Gamma g}{32\pi}(1 - \Gamma^2) \ . \tag{9}$$

These flows are illustrated in Fig. 1. We see that in an isotropic system, $\Gamma = 1$, and the nonlinear coupling g, which embodies the non-equilibrium fluctuations, is relevant. Moreover, for a wide range of anisotropies (namely, all $\Gamma > 0$) the flow is attracted to the isotropic line: the system flows to strong coupling, with emergent rotational symmetry. On the other hand, if the anisotropy is sufficiently strong, so that $\Gamma < 0$, the non-linearity becomes irrelevant and the system can flow to an effective equilibrium state at long scales.

We will now discuss the physics of these two regimes, starting with the isotropic case, which is most relevant to current experiments with polariton condensates.

Isotropic systems – As noted above, rotational symmetry is emergent at long scales if the anisotropy is not too strong at the outset. This is also the regime in which current experimental quantum well polaritons lie. We therefore consider this case first.

On the line $\Gamma = 1$, the scaling of the non-linear coupling $dg/dl = g^2/8\pi$ drives $g \to \infty$; in the growing surface problem the system goes to the "rough" state, with height fluctuations scaling algebraically with length. The analogous behavior in the phase field θ would lead to stretched exponentially decaying order parameter correlations. However, the fact that the phase field is compact implies that topological defects (vortices) in this field exist. Our expectation, based on analogy with equilibrium physics (which admittedly may be an untrustworthy analogy), is that vortices will unbind at the strong coupling fixed point of the KPZ equation. If this happens, it will lead to simple exponential correlations. Testing this expectation will be the object of future work.

We have thus established that the non-linearity, no matter how weak, destroys the condensate at long distances, leading to either stretched or simple exponential decay of correlations throughout the isotropic regime. However, the effects of the nonlinearity only become apparent when g gets to be of order one. Solving the scaling equation we see that this occurs at the characteristic "RG time" $l_* = 8\pi/g_0$; the corresponding length scale is:

$$L_* = \xi_0 e^{\ell^*} = \xi_0 e^{8\pi/g_0} \ , \tag{10}$$

where ξ_0 is the mean field healing length of the condensate. If the bare value of g_0 is small, then the scale L_* can be huge. On length scales smaller than L_*, the system is governed by the linearized isotropic KPZ equation, which, as noted earlier, is the same as an equilibrium XY model. Thus, all of the equilibrium physics associated with two-dimensional BEC, including power law correlations and a Kosterlitz-Thouless defect unbinding transition, can appear in a sufficiently small system.

As parameters, such as the pump power, are changed, the system can lose its apparent algebraic order in one of two ways: (i) the KPZ length L_* is gradually reduced below the system size, or (ii) L_* remains large while the correlations within the system size L are destroyed by unbinding of vortex anti-vortex pairs at the scale L. The latter type of crossover would appear as a KT transition broadened by the finite size. Of course, for any given set of system parameters, a sufficiently large system ($L > L_*$) will always be disordered.

We shall now discuss how the system parameters determine what type of crossover, if any, will be seen in an experiment. We assume that the main tuning parameter is the pump power γ_p, and it will be convenient to track the behavior as a function of a dimensionless tuning parameter $x = \gamma_p/\gamma_l - 1$. But other parameters can also be tuned continuously in an experiment; for example, the effective polariton mass can be varied by illuminating different regions in a spatially varying quantum well. In the supplementary material, we derive the parameters of the KPZ equation for a realistic model of a polariton condensate, described by a two fluid model which includes an upper polariton band, into which polaritons are pumped, and a lower band, where the condensate forms(See the Supplementary Material). The expressions we obtain from that model for the diffusion constants D_α and non-linear coefficients λ_α can be reproduced from those of Eq. (27) for the Ginzburg-Landau model Eq. (1) by the replacement $u_d \to \gamma_l^2/P = u_c\bar{\gamma}/(1+x)$ (cf. Eq. (27) in the Supplementary Material). Here P is the pump rate into the upper band and $\bar{\gamma}$ is a dimensionless parameter characterizing the loss rate (see Supplementary Material). For simplicity we shall also take $K_d = 0$ since this parameter is thought to be small in isotropic two-dimensional quantum wells.

We then have $D = K_c u_c/u_d = K_c(1+x)/\bar{\gamma}$ and $\lambda = -2K_c$. The expression for the noise strength in terms of the two-band polariton model parameters is (see Supplementary Material):

$$\Delta = \frac{u_c\bar{\gamma}}{2x}\left(1 + \frac{(1+x)^2}{\bar{\gamma}^2}\right) . \tag{11}$$

Putting all this together, we obtain an expression for the bare dimensionless coupling constant g_0 in this model, which measures the bare deviation from equilibrium:

$$g_0 = \frac{\Delta\lambda^2}{D^3} = 2\bar{u}\bar{\gamma}^2\left(\frac{\bar{\gamma}^2 + (1+x)^2}{x(1+x)^3}\right). \tag{12}$$

Here $\bar{u} \equiv u_c/K_c$ is the dimensionless interaction constant. Note that g_0 diverges as we approach the mean field transition at $x \to 0^+$, while it decays as $1/x^2$ as $x \to \infty$ at very high pump power.

Hence, at high pump power, the KPZ length scale $L_* = \xi_0 \exp(8\pi/g_0)$ is certainly much larger than any reasonable system size. As the pump power is decreased, and the system approaches the mean field transition at $x = 0$, L_* drops sharply to a microscopic healing length $L^* \approx \xi_0$. L_* drops below the system size when $x \lesssim x_*$, where

$$\frac{x_*(1+x_*)^3}{\bar{\gamma}^2 + (1+x_*)^2} \approx \frac{\bar{u}\bar{\gamma}^2}{4\pi}\ln(L/\xi_0) . \tag{13}$$

For pump powers corresponding to $x > x_*$, the system will appear to be at effective equilibrium, and, hence, may sustain power law order within its confines, whereas for pump power $x < x_*$, the non-equilibrium fluctuations become effective and destroy the algebraic correlations at the scale of the system size. However, it is possible that this crossover at x_* is preceded by unbinding of vortices at values of $x = x_{KT} > x_*$, while the finite system is still at effective equilibrium.

To determine which crossover occurs in a particular system, let us estimate the value of the tuning parameter x_{KT} at which the putative Kosterlitz-Thouless transition *would* occur if the non-linear term λ vanished, or was negligible. Then Eq. (5) obeys a fluctuation-dissipation relation with a temperature set by the noise $T = \Delta$. The KT transition would occur for an equilibrium XY model

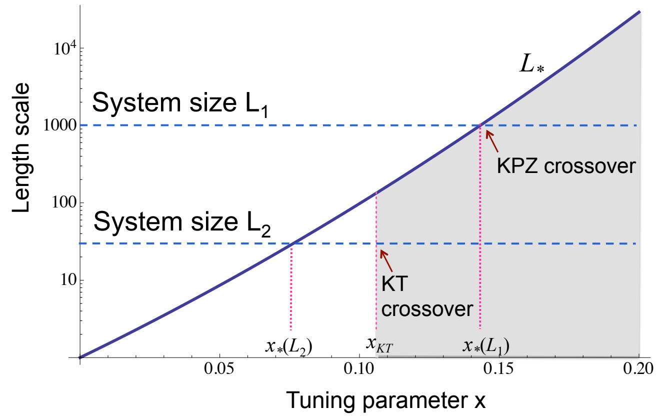

FIG. 2. Dependence of the emergent KPZ length scale L_* (in units of the microscopic healing length) on the tuning parameter $x = \gamma_p/\gamma_l - 1$. This curve was obtained by inserting the expression Eq. (12) for the bare coupling g_0 into Eq. (10) for L_*. While L_* is exponentially large when $x > \bar{u}\bar{\gamma}^2/2\pi$, it goes to a microscopic value ξ_0 at the mean field transition $x = 0^+$. The shaded region marks the scales at which a system would exhibit algebraic correlations. Upon decreasing the tuning parameter x, a finite system will lose its algebraic order in one of two ways: (1) When L_* falls below system size, as in the case of system L_1 shown, or (2) in a KT transition before L_* falls below system size, as in the case of system L_2. Here we have used $\bar{\gamma} = 0.7$, $\bar{u} = 0.5$.

approximately at the point where $\Delta/D = \pi$. Expressing both Δ and D in terms of the tuning parameter x, we obtain the equation for the critical value x_{KT} at which the Kosterlitz-Thouless transition will appear to occur:

$$\frac{x_{KT}(1+x_{KT})}{\bar{\gamma}^2 + (1+x_{KT})^2} \approx \frac{\bar{u}}{2\pi}. \tag{14}$$

We can solve the equations for x_* and x_{KT} in certain simple limits. For example, if we assume weak interactions $\bar{u} \ll 2\pi$, and in addition $(\bar{u}/4\pi)\bar{\gamma}^2\ln(L/\xi_0) \ll 1$, then x_* and x_{KT} are given approximately by $x_* \approx (\bar{u}/4\pi)\bar{\gamma}^2(1+\bar{\gamma}^2)\ln(L/\xi_0)$, and $x_{KT} \approx (\bar{u}/4\pi)(1+\bar{\gamma}^2)$. Under these conditions we expect to see a crossover controlled by vortex unbinding through the KT mechanism, i.e., $x_{KT} > x_*$, if the system size is $L < \xi_0\exp(2/\bar{\gamma}^2)$. For larger system size the crossover will be controlled by the nonlinearities of the KPZ equation. This crossover behavior is summarized in Fig. 2.

Strong anisotropy – If the bare value of the anisotropy parameter is negative $\Gamma < 0$, then the RG equations (9) lead to a fixed point at $g = 0$. Because the non-linear $\lambda_{x,y}$ terms in (5) are irrelevant in this region of parameter space, the linear (and, hence, equilibrium) version of the theory applies. Hence it is possible, for $\Gamma < 0$, to obtain both a power law phase and a KT defect unbinding transition out of it.

The RG flow of the anisotropic KPZ equation for $\Gamma < 0$ was analyzed in Ref. [31] and used to estimate the critical point for vortex proliferation. In principle, we should

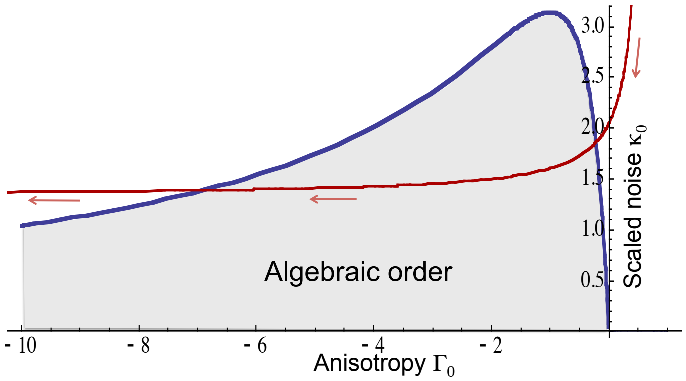

FIG. 3. Phase diagram of a generic anisotropic system exhibiting reentrance. The thin line marks the natural trajectory in an experiment in which only the pump-power is varied. The arrows mark the direction of increasing pump power. Such an experiment will see a reentrant behavior, where the system starts in a disordered state, enters the power-law superfluid and then goes back to a disordered state. Here we've used the two band model of the supplemental materials, with $\bar{\gamma} = 1/2$, $\bar{u} = 2$, $\nu_x = 1/8$ and $\nu_y = 1/4$.

add to these recursion relations terms coming from the vortices. Instead, we will follow reference [31] and assume that the vortex density is low enough that vortices only become important on length scales far longer than those at which the nonlinear effects have become unimportant (i.e., those at which the scaled non-linearity g has flowed to nearly zero). If this is the case, then we can use the recursion relations Eq. (9) for our problem, despite the fact that they were derived neglecting vortices.

Our strategy is then to use those recursion relations to flow to the linear regime, which, as noted earlier, is equivalent to an equilibrium XY model. In this regime, vortex unbinding is controlled by the (bare) parameters $\kappa_0 \equiv \Delta/\sqrt{D_x D_y}$, giving the scaled noise level (cf. Eq. (11)) and the scaled anisotropy Γ_0. The critical point for vortex unbinding can be estimated by solving for the renormalized scaled noise $\kappa(\ell \to \infty) \equiv \kappa(\infty)$ as a function of the bare value using the RG equations of the non-compact KPZ equation; this involves additional recursion relations for D_α and Δ as well as Eq. (9); for details see reference [31]. This gives $\kappa(\infty) = -\kappa_0(1-\Gamma_0)^2/(4\Gamma_0)$. The KT transition occurs at the point where this renormalized value $\kappa(\infty)$ of κ reaches π. Hence the phase boundary in the κ_0-Γ_0 plane is given, within this approximation, by the curve [31]

$$\kappa_0 = -\frac{4\pi\Gamma_0}{(1-\Gamma_0)^2} \ . \tag{15}$$

The assumption in deriving this curve was that the dominant contribution to the stiffness renormalization comes from the non-linear fluctuations, rather than from bound vortex-antivortex pairs, which have been neglected.

There is a broad range of parameters for which a system enters a regime $\Gamma_0 = (D_x\lambda_y)/(D_y\lambda_x) < 0$, in which true power-law order and a KT transition exist. A natural and simple model of such a system is the "two-band polariton model" discussed earlier, and treated in the supplemental material. Two important parameters determining the behavior of the anisotropic system are the ratios of the dissipative to coherent phase stiffnesses along the two directions, $\nu_\alpha = K_d^\alpha/K_c^\alpha$. Using the expressions for D_α, λ_α, and Δ in the two-band model we can easily obtain the dependence of the bare anisotropy parameter Γ_0 and the bare scaled noise κ_0 on the tuning parameter x:

$$\Gamma_0 = \frac{[\nu_y(1+x)-\bar{\gamma}]\,[\nu_x\bar{\gamma}+1+x]}{[\nu_x(1+x)-\bar{\gamma}]\,[\nu_y\bar{\gamma}+1+x]}, \tag{16}$$

$$\kappa_0 = \frac{\bar{u}}{2x}\,\frac{[\bar{\gamma}^2+(1+x)^2]}{\sqrt{[\nu_y\bar{\gamma}+(1+x)][\nu_x\bar{\gamma}+(1+x)]}}. \tag{17}$$

Now consider gradually increasing the pump power, and hence x, from the mean field threshold $x = 0$. We notice that if the system parameters obey $\bar{\gamma} > \nu_y > \nu_x$, then $\Gamma_0(x)$ starts out positive at $x = 0$, is reduced to negative values as x is increased past $x = \frac{\bar{\gamma}}{\nu_y} - 1$ and eventually runs off to $\Gamma_0 = -\infty$ at a finite value of x (namely, $x = \frac{\bar{\gamma}}{\nu_x} - 1$). If at the same time $\bar{u}$ is sufficiently small, then the experimental trajectory in the $\kappa_0 - \Gamma_0$ plane is guaranteed to cross the dome marking the condensate (algebraic order) phase as determined in Eq. (15). The condition on $\bar{u}$ for this crossing to occur is

$$\bar{u} < 2\pi\frac{(\bar{\gamma}-\nu_y)}{\bar{\gamma}\nu_y(1+\nu_y^2)} \ . \tag{18}$$

Thus, we not only naturally achieve the ordered phase in this anisotropic system by varying the driving, but we do so in a *reentrant* manner: we enter the phase, and then leave it again, as the driving is increased. Note that the analysis for $\gamma > \nu_y > \nu_x$, is the same if we take $\Gamma \to 1/\Gamma$.

Note, finally, that both the ordered phase and reentrance are possible in a wide range of microscopic models; we have focused on this particular one both because it is believed to be a reasonably accurate description of a polariton condensate, and because it nicely demonstrates the possibilities.

Three dimensional systems– Nothing in our derivation of the KPZ equation was restricted to two spatial dimensions[32]. In three dimensions, a renormalization group (RG) analysis of the anisotropic KPZ equation (5) shows that the non-linear $\lambda_{x,y}$ terms are irrelevant, in the RG sense, as long as they are weak enough. This in turn implies that the correlations of the system behave at long wavelengths as an equilibrium three-dimensional XY model; in particular the system can have true long range order. Such a state would disorder through the dynamical phase transition described in Ref. [8].

However there is another possible transition out of the ordered phase. In three dimensions, the isotropic KPZ equation is believed to have an unstable fixed point at intermediate coupling [33]. That is, if the bare nonlinearities measuring the departure from equilibrium are strong enough, they do become relevant, leading to divergent height fluctuations. This implies that the three-dimensional driven BEC problem may have another disordered phase, in which order parameter fluctuations are short-ranged, but topological defects are still bound. Such a topologically ordered phase would have a non-zero, wavevector-dependent "anomalous" superfluid density $\rho_s(\mathbf{q}, \omega)$ that vanished as a power law in wavenumber $\mathbf{q}$ or frequency ω as $\mathbf{q}, \omega \to 0$.

In addition, order parameter ψ correlations would decay as stretched exponentials, rather than the simple exponential decay that would occur in the fully disordered phase with unbound vortices. This might be a difficult criterion to use experimentally, however, since stretched exponentials can look very similar to ordinary exponentials; both, for example, have finite Fourier transforms as $\mathbf{q} \to \mathbf{0}$. Perhaps the best way to detect such a strong coupling phase would be to detect singular behavior in observable quantities (e.g., total polariton density) as the system goes through the vortex unbinding transition.

Such a seemingly paradoxical coexistence of short-ranged order and non-zero stiffness also occurs in disordered superconductors [36] and liquid crystals in aerogel [37], but this would be a rare example of such a phase occurring in a system *without* quenched disorder. This will be discussed in more detail in a future publication.

Conclusions – We have shown that quasi-long-ranged polaritonic Bose Einstein Condensation cannot exist in a two-dimensional system, unless the system is strongly anisotropic. This result is obtained by mapping of the condensate dynamics to the (compact) anisotropic Kardar-Parisi-Zhang equation. Non-equilibrium fluctuations generated by the drive translate to the non-linear term in the KPZ equation, which disorders the condensate at long scales. We remark that earlier work, which predicted long range algebraic order in two-dimensional driven condensates [34], relied on a linear (Bogoliubov) theory, which may appear on intermediate scales but, as our analysis shows, is invalidated at long distances due to the relevant non-linear term.

It is worth noting that recent analysis of the dissipative Ginzburg-Landau equation without noise found that the mean field condensate is destroyed, through a completely different mechanism, by static disorder [35]. The consequences of the interplay of noise and the static disorder is an interesting question for future investigations.

We have also shown that for sufficiently strong breaking of four- or six- fold rotational symmetry, the nonlinearity becomes irrelevant, allowing the establishment of effective equilibrium. In this case, the system can develop algebraic correlations through a standard Kosterlitz-Thouless transition.

Our analysis can be extended to three dimensions, where it predicts a true Bose condensate which may be established through a standard thermodynamic transition, in agreement with earlier results [8]. However, the analysis may also allow for a different, non-equilibrium transition controlled by a strong coupling fixed point of the three dimensional KPZ equation. Disordering of the condensate in this way might give rise to a topologically ordered phase: short-range ordered, but distinct from the usual uncondensed state in that vortex loops do not proliferate.

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Supplementary Material

Polariton condensate model with reservoir

In the main text we worked with a Ginzburg-Landau model including only the lower polariton band. Such a model clearly gives the correct universal physics. However, in order to find how the parameters of the anisotropic KPZ equation change as actual experimental parameters are varied requires to start from a more microscopic model of the polariton degrees of freedom.

The standard model for describing these systems is a "two fluid" model which includes the particles in the "upper-polariton band" acting as a reservoir with local density n_R for the condensate which forms in the "lower polariton band" [3]. Here we generalize the model slightly in order to include dissipative mass terms and anisotropy:

$$\begin{aligned}\partial_t\psi &= \left[\sum_{\alpha=x,y}(iK_c^\alpha + K_d^\alpha)\partial_\alpha^2 - ir_c - \gamma_l - iu_c|\psi|^2 + Rn_R\right]\psi + \zeta,\\ \partial_t n_R &= P - Rn_R|\psi|^2 - \gamma_R n_R,\end{aligned} \tag{19}$$

where $\langle\zeta^*(\mathbf{x},t)\zeta(\mathbf{x}',t')\rangle = 2\sigma\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$. It is usually assumed that the reservoir relaxation time γ_R is faster than all other scales. Hence we may solve the reservoir density independently assuming it is time independent

$$n_R = \frac{P}{\gamma_R + R|\psi|^2}. \tag{20}$$

Substituting this in the equation for ψ we obtain

$$\partial_t\psi = \left[\sum_\alpha(iK_c^\alpha + K_d^\alpha)\partial_\alpha^2 - ir_c - \gamma_l - iu_c|\psi|^2 + \frac{P}{\eta+|\psi|^2}\right]\psi + \zeta, \tag{21}$$

where we have eliminated R and γ_R for the single parameter $\eta = \gamma_R/R$. We note that the amplitude of the white noise is given by the total loss rate (γ_l) and gain, and since in steady state the loss and gain must be equal we simply have $\sigma = \gamma_l$ in this case.

In the following, as in the main text we work in the phase-amplitude representation $\psi(\mathbf{x},t) = (M_0 + \chi(\mathbf{x},t))e^{i\theta(\mathbf{x},t)}$ and expand around the homogeneous mean field solution. Let us therefore first solve for the mean field steady state. The real part of the equation gives $\gamma_l = P/(\eta + M_0^2)$ from which we can deduce the condensate density $M_0^2 = P/\gamma_l - \eta$. The imaginary part of the equation is $r_c = -u_c M_0^2$. It is also worth noting that loss comes only from the term γ_l, since there is no two-particle loss term in this model (instead saturation is reached due to the non-linear reduction of the pump term). Hence in steady state, when loss is equal to gain, the noise term is simply $\sigma = \gamma_l$.

We now proceed to write the equations of motion for χ and θ to linear order in χ. This gives

$$\begin{aligned}M_0^{-1}\partial_t\chi &= -2\gamma_l^2P^{-1}M_0\chi - K_c^\alpha\partial_\alpha^2\theta - K_d^\alpha(\partial_a\theta)^2 + M_0^{-1}\mathrm{Re}\zeta,\\ \partial_t\theta &= -2u_c\chi + K_d^\alpha\partial_\alpha^2\theta - K_c^\alpha(\partial_\alpha\theta)^2 - M_0^{-1}\mathrm{Im}\zeta.\end{aligned} \tag{22}$$

Now as in the main text we can eliminate χ to obtain the KPZ equation for θ, where $\alpha = x,y$ is summed over and

$$\partial_t\theta = D^\alpha\partial_\alpha^2\theta + \frac{1}{2}\lambda^\alpha(\partial_\alpha\theta)^2 + \bar\zeta, \tag{23}$$

where

$$\bar\zeta = M_0^{-1}\left(\mathrm{Re}\zeta - \frac{u_cP}{\gamma_l^2}\mathrm{Im}\zeta\right). \tag{24}$$

The noise parameter in $\langle\bar\zeta^*(\mathbf{x},t)\bar\zeta(\mathbf{x}',t')\rangle = 2\Delta\delta(\mathbf{x} -$

$\mathbf{x}')\delta(t-t')$ is here given by:

$$\Delta = \frac{\gamma_l^2/2}{P-\eta\gamma_l}\left(1+\frac{u_c^2P^2}{\gamma_l^4}\right) = \left(1+\frac{u_c^2\eta^2\gamma_p^2}{\gamma_l^4}\right)\frac{\gamma_l^2/2\eta}{\gamma_p-\gamma_l}$$
$$= \frac{u_c\bar{\gamma}}{2x}\left(1+\frac{(1+x)^2}{\bar{\gamma}^2}\right), \tag{25}$$

where we have defined $\gamma_p \equiv P/\eta$, the dimensionless tuning parameter $x = \gamma_p/\gamma_l - 1$ and the dimensionless loss parameter $\bar{\gamma} \equiv \gamma_l/(\eta u_c)$. Below we will also need the dimensionless interaction strength $\bar{u} \equiv u_c/\sqrt{K_c^x K_c^y}$ and the ratios $\nu_\alpha = K_d^\alpha/K_c^\alpha$. The parameters of the anisotropic KPZ equation may now be written as:

$$D_\alpha = K_c^\alpha\left(\frac{K_d^\alpha}{K_c^\alpha}+\frac{u_cP}{\gamma_l^2}\right) = K_c^\alpha\left(\nu^\alpha+\frac{1+x}{\bar{\gamma}}\right), \tag{26}$$
$$\lambda_\alpha = 2K_c^\alpha\left(\frac{K_d^\alpha u_cP}{K_c^\alpha\gamma_l^2}-1\right) = 2K_c^\alpha\left(\nu^\alpha\frac{1+x}{\bar{\gamma}}-1\right).$$

As noted in the main text, the expressions for the diffusion constants D_α and non-linear coefficients $\lambda_\alpha$can be obtained from the predictions Eq. (27) for the Ginzburg-Landau model Eq. (1), if we make the replacement

$$u_d = \frac{\gamma_l^2}{P} = \frac{u_c\bar{\gamma}}{1+x}. \tag{27}$$

Crossover scales in isotropic polariton condensates

We will now use the results just presented for the isotropic case without dissipative mass terms; i.e., $\nu_x = \nu_y = 0$ (for the anisotropic case, see main text). This implies $D = K_c(1+x)/2\bar{\gamma}$ and $\lambda = -K_c$. The dimensionless coupling constant g is then given by

$$g = \frac{\Delta\lambda^2}{D^3} = 2\bar{u}\bar{\gamma}^2\left(\frac{\bar{\gamma}^2+(1+x)^2}{x(1+x)^3}\right). \tag{28}$$

From the expression for g we can extract the dependence of the KPZ length on the tuning parameter:

$$\log(L_*/\xi_0) = \left(\frac{4\pi}{\bar{u}\bar{\gamma}^2}\right)\frac{x(1+x)^3}{\bar{\gamma}^2+(1+x)^2}. \tag{29}$$