Spread of correlations in long-range interacting systems

P. Hauke$^{1,2}$ and L. Tagliacozzo$^3$

$^1$Institute for Quantum Optics and Quantum Information of the Austrian Academy of Sciences, A-6020 Innsbruck, Austria
$^2$Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria
$^3$ICFO-Institut de Ciencies Fotoniques, Av. Carl Friedrich Gauss, 3, 08860 Castelldefels, Barcelona, Spain.

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Understanding the dynamics of many-body systems is crucial for understanding, e.g., thermalization or transmission of information. Nevertheless, little is known in the case of quantum systems with long-range interactions. Here, we analyze the long-range Ising model in a transverse field, where interactions decay as a power-law with distance $r^{-\alpha}$, $\alpha > 0$. Using complementary numerical and analytical techniques, we identify three dynamical regimes: short-range-like with an emerging light cone for $\alpha > 2$; weakly long-range for $1 < \alpha < 2$ without a clear light cone but with a finite propagation speed of excitations; and fully non-local for $\alpha < 1$ with instantaneous transmission of correlations. This last regime breaks generalized Lieb–Robinson bounds. Numerical calculation of the entanglement spectrum demonstrates that the usual picture of propagating quasi-particles remains valid for long-range interactions. This allows an intuitive interpretation in terms of qualitative changes to the spin-wave dispersion, leading to diverging quasi-particle velocities in the long-range regime. Our results may be tested in state-of-the-art trapped-ion experiments.

Recent years have seen a surge of interest in the dynamics of quantum systems. One reason is that clean quantum dynamics can now be observed in finely controlled physical setups, such as ultracold atoms $^{[1,3]}$. Other experiments in this direction can be expected in the near future in polar molecules $^{[4–6]}$ and Rydberg atoms $^{[7,8]}$, or with trapped ions $^{[9–14]}$, where first steps towards studying dynamics have already been done $^{[14]}$. It is currently not clear, however, how the long-range interactions in these cases influence the dynamics.

From the theoretical side, one seeks to understand the long-time behavior of out-of-equilibrium many-body systems, especially driven by the question how thermalization can emerge from unitary quantum dynamics. The hope is that, after long times, a small region would notice the presence of the rest of the system as if it was in contact with a thermal bath and would essentially equilibrate to a thermal state. We already know that not all systems equilibrate to thermal states $^{[15]}$, which immediately calls for a more general classification of possible equilibrium states. For example, it is now widely accepted that integrable models equilibrate locally towards Generalized Gibbs states $^{[16]}$.

The typical scenario for such studies are global quenches, where an initial state is evolved under a Hamiltonian whose eigenstates are only locally different from the initial state. Such evolutions (for short-ranged systems), as a consequence of the Lieb–Robinson bound, generate much less entanglement $^{[23,24]}$ than those after global quenches, and are thus treatable with TN techniques.

A simpler class of out-of-equilibrium dynamics involves local quenches, where a system evolves under a Hamiltonian whose eigenstates are only locally different from the initial state. Such evolutions (for short-ranged systems), as a consequence of the Lieb–Robinson bound, generate much less entanglement $^{[23,24]}$ than those after global quenches, and are thus treatable with TN techniques.

The above-mentioned Lieb–Robinson bound has proven essential for understanding the complexity of quantum states $^{[25,26]}$. It applies to systems with sufficiently short-range interactions, allowing to formulate a variety of general theorems, e.g., connecting excitation gaps and decays of correlations $^{[27,28]}$. In this article, we want to address the question how the Lieb–Robinson bound breaks down in long-range interacting systems.

In its essence, the Lieb–Robinson bound formulates the principle of causality by predicting a finite speed at which correlations can spread. In this way, one can identify a ‘light cone’ outside of which correlations are exponentially suppressed. Mathematically, under certain assumptions, the Lieb–Robinson bound can be expressed as a bound for the time-dependent commutator between two operators $O_A, O_B(t)$, defined at $t = 0$ on two disjoint regions of the system $A$ and $B$ separated by a distance $L$ $^{[25,26]}$.

$$[O_A, O'_B(t)] \leq ||O_A|| ||O'_B|| g(L) v_L t,$$  \hspace{1cm} (1)
where on the right hand side the norm is the operator norm, $v$ the Lieb–Robinson velocity, and $g(L)$ an exponentially decaying function.

The Lieb–Robinson bound has important implications for thermalization: if the system locally equilibrates to a Generalized Gibbs ensemble, also time-dependent correlation functions can be described by the same ensemble \[20\]. In some systems, the Lieb–Robinson bound can be understood using an intuitive pseudo-particle picture. This applies if the low-lying excitations can be obtained by populating (for translational invariant systems) different pseudo-particle momentum states, with the vacuum characterized by the absence of pseudo-particles. Then, the system responds to a local perturbation by emitting pseudo-particles propagating at different finite speeds. The fastest particles, the ones defining the causal cone, propagate at a speed that is often identified as the Lieb–Robinson velocity for that specific model.

Much less is known about how correlations spread in the presence of long-range interactions, although these become important in many different contexts. Whenever one has a local model where some of the constituents propagate much faster than the others, one can encode the presence of the fast constituents in an effective description of the slow ones involving a non-local interaction. A prime example is Quantum Electrodynamics, describing the contact interaction between photons propagating at the speed of light and charges. In the non-relativistic limit, where the charges move much slower than the light, the presence of photons can be encoded in a long-range Coulomb potential between the charges. Theories with long-range interactions can have over-extensive energies \[30, 31\] and are thus strongly non-local. In such circumstances, one would expect that concepts like causality and the locality of quasi-particle excitations should be reconsidered.

The purpose of this manuscript is to address this issue on a theoretical level, using complementary analytical and numerical calculations. We find three different dynamical regimes, with a break-down of Lieb–Robinson bounds at strong long-range interactions. We are able to explain these regimes through the above-mentioned pseudo-particle picture. Finally, we discuss experimental regimes in trapped-ion setups where our findings can be observed.

In this article, we study the out-of-equilibrium dynamics generated by a long-range interaction in the simplest possible scenario that can be implemented in trapped-ions experiments \[32\], namely the long-range transverse Ising chain (LRTI)

$$H = \sum_{\langle i,j \rangle} \sin(\theta) \sigma_i^x \sigma_j^z + \cos(\theta) \sum_i \sigma_i^z.$$  

(2)

Here, $\sigma$ denote the usual spin-1/2 Pauli matrices, and we set the fundamental energy unit \[33\] and the lattice spacing to unity. We consider a finite chain of $L$ sites with open boundary conditions. The parameter $\alpha$, responsible for the long range of interactions, is varied within

![FIG. 1. (Color online) (Non-)light cones. (a-c) Block entanglement entropy $\Delta S_i = S_i(t) - S_i(0)$ from TDVP ($\theta = \pi/5$, $L = 100$). (d-f) Polarization $\delta m_i = \langle S_i^z \rangle + 1/2$ from LSWT ($\theta = \pi/20$). For $\alpha > 2$, the excitation spreads light-cone like, as in the short-range model. For $2 > \alpha > 1$, there is no well-defined wave front, but the excitation needs a finite time to bridge large distances. For $\alpha < 1$, the excitation spreads immediately over the entire system. Black dashed lines in (d-f) denote the maximal spin-wave group velocity. In (f), they practically coincide with the abscissa.](image)
that the two methods provide compatible results, showing that the time evolution they describe is essentially semi-classical. This agreement is reasonable, given that $|\psi_0\rangle$ contains a single excitation. Since the excitation density spreads during the evolution, the assumption of non-interacting quasi-particles that underlies the LSWT improves over time. Therefore, LSWT predictions should be quantitatively correct over all time scales.

Numerical results— We now study how an excitation, initially localized at the chain center, spreads. For TDVP we choose $\theta = \pi/5$, which is not accessible with LSWT because here magnetic long-range order is strongly reduced due to a nearby quantum phase transition. In Fig. 1, we study the spread of quantum correlations through the block entropy $S_l = -\sum_n \rho_n^l \log \rho_n^l$, where $\rho_n^l$ is the $n$-th eigenvalue of the reduced density matrix $\rho_l$ involving the spins $1, \ldots, l$.

As known from [34], in the ground state of the $z$-polarized phase of the LRTI, the long-range interactions cause $S_{l/2} \propto \log L$ for $\alpha < 2$. Therefore, to isolate the growth of the entropy generated during the time evolution, we analyze the excess of block entropy with respect to the initial state, $\Delta S_l = S_l(t) - S_l(0)$. We have performed the TDVP simulations with an algorithm discussed in the Appendix that generalizes the ones available in the literature [34]. We consider chain sizes up to $L = 150$, and we have checked that the accuracy of MPS with matrix sizes $\chi \leq 200$ is sufficient.

For LSWT, we exemplify the resulting dynamics for $\theta = \pi/20$ (Fig. 1f), where the ground state is almost perfectly polarized, $\langle S_l^z \rangle \approx -1/2$ [34]. In this case, a useful measure for the spread of the perturbation is the excess magnetization $\delta m_z = \langle S_l^z \rangle + 1/2$. Notably, within LSWT, this gives a direct measure for the single-site entanglement entropy, $S_l^{(1)} = (\delta m_z + 1) \log(\delta m_z + 1) - \delta m_z \log \delta m_z$ [11, 12]. Figure 1 evidences the similar behavior for the two methods and the two $\theta$ regimes.

For generic $\theta$, we identify three dynamical regimes as a function of $\alpha$. (i) For $\alpha \geq 2$ [realized in Nature, e.g., for van-der-Waals ($\alpha = 6$) or dipole–dipole ($\alpha = 3$) interactions], the system behaves as if short-range interacting, with the spread of the local perturbation essentially bounded by a light-cone. The excitation maximum defines a clear wave front, and its linear propagation gives a constant Lieb–Robinson velocity as in the short-range model, coinciding with the maximal spin-wave group velocity. Outside the light cone, correlations decay algebraically with a power determined by $\alpha$, thus obeying the generalized Lieb–Robinson bounds. (ii) In the range $2 > \alpha > 1$, although at short times there appears an effect resembling a light cone, it does not really bound the propagation of the perturbation, since correlations consistently leak out of it. Further, the propagation of the perturbation shows complex interference effects due to longer-range spin flips, and at sufficiently large times one cannot identify an unambiguous wave front. Still the excitation needs a finite time to bridge larger distances. (iii) For $\alpha < 1$ (typically, $\alpha = 1$ corresponds to Coulomb- or gravitation-like potentials), the generalized Lieb–Robinson bounds valid for $\alpha > 1$ can no longer be defined. As a consequence, the system becomes truly long ranged, and correlations spread practically instantaneously over the entire chain.

These results complement the one available about thermalization in disordered systems with random interactions that are modulated by a long-range power law. In that case, the time average of local observables tends to a value predicted by a Generalized Gibbs ensemble only if $\alpha < 1$ [43].

Our findings differ from previous results for the specific cases of Hamiltonians made of mutually commuting terms, such as, e.g., Eq. (2) with $\theta = \pi/2$. In such settings, at $\alpha = 1$ nothing special seems to happen, while the value $\alpha = 0.5$ seems to separate two dynamical regimes [44], in one of which one finds, e.g., prethermalization plateaus [45]. Also, during time evolution with commuting Hamiltonians, the block entropy of subsystems can increase unchecked with block size for $\alpha \leq 0.5$, whereas for $\alpha > 1$ it is strictly upper bounded [46].

Pseudo-particle dispersion relation— The qualitatively different behavior in the regimes (i-iii) can be understood in a simple quasi-particle picture: During the local quench, all spin-wave $k$-modes become approximately equally populated (occupation $\approx 1/L$). If the pseudo-particles do not interact (a good approximation for low pseudo-particle density), they will subsequently propagate with the group velocity corresponding to their $k$ value, $v_g = \partial \omega_k / \partial k$, which depends only on the dispersion relation $\omega_k$ (see Fig. 2; see Appendix for an analytical formula from LSWT).

In the range $2 > \alpha < \infty$, the maximal group velocity $v_{\text{max}}$ is achieved around $k = \pi$, and does barely de-
pend on system size or $\alpha$ (Fig. 2, top). At $\alpha < 2$, however, $\omega_k$ acquires a cusp at $k = \pi$, where $|v_\rho|$ diverges like $|k - \pi|^{-2}$. Consequently, $v_{\text{max}}$ is attained at $k = \pi \pm 2\pi/L$ and diverges as $v_{\text{max}} \propto (2\pi/L)^\alpha - 2$.

Still, the time scale in which pseudo-particles can reach the boundary, $t_b \equiv L/(2v_{\text{max}})$, scales as $L^{\alpha - 1}$, which diverges for $1 < \alpha \leq 2$; the time to reach the boundary increases with system size, even for the fastest mode.

The long-range effects become more dramatic at $\alpha < 1$, with a stronger divergence $v_\rho k \propto (\alpha - 3)/2$. Here, for the fastest mode, $t_b$ decreases with system size (in fact, even for a diverging number of modes, see Fig. 2 and the Appendix). In Fig. 2, the transition between the three regimes can be clearly identified.

The spin-wave dispersion also explains the diffusive effect encountered at small $\alpha$ (see Fig. 1a-f). With decreasing $\alpha$, the dispersion becomes flatter around the sides of the Brillouin zone. Therefore, there are many slow quasi-particles that remain in the central region for a long time, giving rise to an apparent diffusive core of high density.

**Scaling of entanglement entropy**—To numerically confirm the validity of the pseudo-particle picture, we analyze within the TDVP the scaling of the EE of half of the chain $S_{L/2}(t)$ during the time evolution. Interestingly, for all values of $\alpha$ considered, the excess entropy $\Delta S_{L/2}(t)$ initially increases as a power of $t$ and then saturates to a value very close to $\Delta S_{L/2}(t) = \log 2$, independent of system size (Fig. 3c). The initial growth is faster for smaller $\alpha$, in agreement with the presence of faster pseudo-particles. Before entering the saturation regime, however, systems with smaller $\alpha$ start to evolve slower, in agreement with the appearance of a diffusive evolution. The fact that the excess of EE of a block saturates to a value independent of its size is in remarkable contrast to the ground-state properties. This effect finds a natural explanation in the semi-classical picture of pseudo-particles: the states that dominate the time evolution are states with only one pseudo-particle; the log 2 is then immediately understood as coming from the pseudo-particle picture corresponding to the pseudo-particle being in the left or the right part of the chain. The other eigenvalues are significantly smaller, as a confirmation of the goodness of semi-classical descriptions of the evolution.

A further confirmation comes from the half-chain entanglement-spectrum evolution, $h_{0n}(L/2, t) = \log \rho_{L/2}^{0n}(t)$, where $\rho_{L/2}$ is the $n$-th eigenvalue of the reduced density matrix of half of the chain. As seen in Fig. 3b, the spectrum is dominated by only few eigenvalues, with two of order one as expected from the log 2 asymptote, and a huge number of eigenvalues below $10^{-5}$.

These eigenvalues grow steadily, but we expect that they do not affect equilibrium properties, since they are associated to higher energies and thus, at long times, their effect should average out. These findings are in agreement with similar observations in short-range systems \[24\] [45], where semi-classical models provided a good description of these kinds of out-of-equilibrium dynamics.

**Experimental implementation**—Finally, let us remark that our system does not break non-locality of the physical world, as one should hope. E.g., in the trapped-ion implementation, Hamiltonian (2) describes an effective dynamics for electronic states of the ions, which are coupled by collective phonon modes by employing laser fields \[32\] [39] [50]. The phonon dynamics can be neglected on time scales much larger than those associated to the detuning between laser driving and phonon frequencies. These time scales are typically $\mathcal{O}(10\mu s)$. Moreover, the derivation of Eq. (2) employs a rotating-wave approximation in the phonon frequencies, corresponding to neglecting terms that average to zero on time scales $\mathcal{O}(1\mu s)$. When the group velocity reaches these time scales, the effective Hamiltonian (2) breaks down, just as how the Coulomb potential is no longer valid when charged particles move close to the speed of light. On the other hand, the time scale of the spin interactions is typically $\hbar/J = \mathcal{O}(1\mu s)$. Therefore, although the group velocities of the spin system cannot truly diverge, they can be several times larger than the scale set by $J$, thus still providing a drastic effect. Therefore, in typical practical implementations \[9\] [14], it should be possible to explore the three widely different regimes discussed above.

In typical experiments, the system size is finite, but the three regimes are distinguishable already in small chains. In the short-range regime (i), the ratio of wave-front maximum and the subsequent minimum increases with system size, thus defining an increasingly sharp wave front. Contrarily, in the weak long-range regime (ii), this ratio decreases with system size, until the wave-front maximum disappears, making it impossible to unambiguously define a wave front. In the strong long-range regime (iii), the excitations start to reach the boundary of the system.

![Fig. 3](image-url) (Color online) (a) Growth of the entanglement entropy. The excess $\Delta S_{L/2}(t)$ grows initially as a power law with $t$ for all considered $\alpha$. It then saturates to log 2 independent of system size, as expected from the pseudo-particle picture. This is shown in the insets where we compare the saturation value for chains of different length and for completeness show that there is no residual dependence of the saturation value on the MPS matrix dimension $\chi$. (b) Evolution of the entanglement spectrum. The entanglement spectrum is dominated by only two eigenvalues, which in the pseudo-particle picture correspond to the pseudo-particle being in the left or the right part of the chain. The other eigenvalues are significantly smaller, as a confirmation of the goodness of semi-classical descriptions of the evolution.
immediately, independent of system size.

Conclusions—We have numerically identified three qualitatively different regimes of the dynamics in the LRTI model, and we have reached an intuitive understanding through analytical studies of the spin-wave dispersion relation. The validity of such studies is backed by the agreement of LSWT results with results extracted with the quasi-exact TDVP in the short-time regime.

We have found qualitative changes of the spin-wave dispersion relation at $\alpha = 2$ and $\alpha = 1$, indicating the transition from short-range to weakly long-range to strongly long-range physics. In the last case, a strong divergence of the quasi-particle velocities leads to a practically instantaneous spread of excitations through the system. It will be extremely interesting to study how this effect influences thermalization and how our findings carry over to more complicated models or other dimensionalities.

Finally, we have discussed the regime where strong long-range physics should appear in typical trapped-ion experiments, and outlined a way to observe it in small chains. We hope our findings to inspire experiments along these lines.

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Appendix A: Time-reversal-symmetric scheme for the TDVP applied to long-range Hamiltonians

In a previous work, it has been shown how to find an MPS approximation to the ground state of a long-range Hamiltonian using an extension of the time-dependent variational principle (TDVP) [31]. The TDVP is an algorithm that uses the geometric notion of the MPS tangent plane [35, 40]. It allows to find the ground-state description of a MPS by solving a differential equation for the tensors defining the MPS. The same generalization of the TDVP presented in [34] can be used to perform real-time evolutions [35, 40] for systems with long-range interactions, which we have exploited to compute the quench dynamics described in the main text. The main difference when performing the dynamics – instead of imaginary time as is necessary to obtain the ground state – in real time, is that special care has to be taken to ensure that the algorithm does not violate time-reversal symmetry. This immediately ensures that the algorithm conserves both the norm and the energy of the initial state, as it should. Care is advised, as real-time evolution does not enjoy the same self-correction as imaginary-time evolutions where small errors in one step can be corrected in the next step. Here, instead, any small error is propagated along the simulation, and one needs further care to minimize those.

Here, we describe the steps we follow in order to ensure the time-reversal invariance of the integrators scheme that we apply to the above-mentioned differential equations. This is a simple modification of the technique proposed in [35, 41], suitable for evolutions dictated by long-range Hamiltonians in finite chains.

First, we briefly recall the general strategy that is common to both real-time and imaginary-time dynamics.

(1) Encode the starting state of the time evolution as a MPS described by the set of tensors $L \{ A_n \}, A_n, n = 1 \cdots L |\psi_0(A)\rangle$ and (2) encode the long-range Hamiltonian as a MPO described by a set of tensors $L \{ O \}, H(\{ O \})$. The evolved state is obtained by (3) solving the Schrödinger equation for a short time interval $dt$ with initial condition given by $|\psi_0\rangle$,

$$i\partial_t |\psi(\{A(t)\})\rangle = H(\{ O \})|\psi(\{A(t)\})\rangle$$, \hspace{1cm} (A1)

where we set $h = 1$. (4) To solve the above equation inside the manifold of MPS with fixed tensor dimensions, one needs to introduce tangent vectors. These are generally defined through two sets of tensors $\{ A \}$ and $\{ B \}$, and are expressed as the linear combination of MPS defined by $A$ everywhere but one $B_n$ at a specific place, in formula

$$|T(\{ A \}, \{ B \})\rangle = \sum_n |\psi(\{A\}_{n}, B_n)\rangle$$, \hspace{1cm} (A2)

where we have used the notation $\{ A \}_{n}$ to define the set of $A$ tensors from which we have removed tensor $A_n$.

(5) While the left hand side of the above equation $|A|$ defines a tangent space to the manifold of the MPS with fixed bond dimension, the right hand side is not contained in that space and should be explicitly projected onto it. In formula, we would like to find the tangent vector $|T|$ that minimizes the distance from $H |\psi(\{A(t)\})\rangle$,

$$|T(\{ A \}, \{ B^* \})\rangle, : \min \| |T| - H |\psi(\{A(t)\})\rangle \|^2.$$ \hspace{1cm} (A3)

In practice, in the canonical form, the computation is simplified by requiring that the tangent vectors are orthogonal to the original vector. To ensure the orthogonality, the $B_n$ tensors in the tangent vectors are defined as the contraction of auxiliary tensors, (for normalization convenience) the inverse square root of the reduced density matrix, times a matrix of free coefficients of dimension called $X_n$, and a fixed projector $V_n$ on the orthogonal space to the one on which the starting vector is defined.

(6) At this point, one can discretize Eq. (A1) and integrate it iteratively through

$$A_n(t + dt) = A_n(t) + i dt (B_n)^*$$, \hspace{1cm} (A4)

for all sites $n = 1 \cdots L$.

We now turn to real-time dynamics, and we focus specifically on designing a time-reversal-invariant integrator scheme. This requires improving the first-order
integrator \( A_n \) to at least the so-called middle-point integrator. This involves finding an intermediate step for each \( n \), \( A_n(t + dt/2) \) such that both \( A_n(t + dt) = A_n(t + dt/2 + dt/2(B_n)^*) \) and \( A_n(t) = A_n(t + dt/2) - idt/2(B_n)^* \), where \( \{ \tilde{B} \} \) is the set of tensors defining the projection onto the tangent space of the action of the Hamiltonian on the state \( |\psi(t + dt/2)\rangle \). The two above conditions can be taken as a definition of \( A_n(t + dt/2) \) that we then use to complete the evolution step by just integrating the state for another \( dt/2 \),

\[
A_n(t + dt) = A_n(t + dt/2) + idt/2(B_n)^*,
\]

so that we ensure that the evolution is invariant under time reversal. The important part becomes finding the intermediate \( A_n(t + dt/2) \). As suggested in [38], one can devise an iterative procedure to determine \( A_n(t + dt/2) \). Here, we describe an alternative procedure to the one presented in [38] that is well suited for finite-chain Hamiltonians encoded in MPO as the ones discussed in this paper. The procedure consists in proceeding locally in the evolution (site by site) requiring that each local step is time-reversal invariant. For this reason, chosen a position \( n \) in the chain, one proceeds by

1. Obtaining a trial \( A_n^0(t + dt/2) \) by solving Eq. (A1) for a time step \( dt/2 \), with the initial state locally described by \( A_n(t) \).
2. Obtaining a trial \( B \) by finding the best tangent vector that approximates the r.h.s. of Eq. (A1).
3. Evolving back \( A_n^0(t + dt/2) \) to \( A_n^0(t) \) (that initially will differ from \( A_n(t) \)) by solving Eq. (A1) for a time step \( -dt/2 \), with an initial state locally described by \( A_n(t + dt/2) \).
4. Compute the \( \Delta A_n^0(t) = A_n(t) - A_n^0(t) \) and project it onto the tangent space defined at \( A_n^0(t + dt/2) \).
5. Compute the error as \( E_n^t = \sqrt{||\psi(\{ A \}, \Delta A_n^0(t))||^2} \).
6. In this way, we can obtain the improved estimate of the middle point \( A_n(t + dt/2) \) as \( A_n^0(t + dt/2) = A_n^0(t + dt/2) + P_{\tilde{B}_n} \Delta A_n^0(t) \), where \( P_{\tilde{B}_n} \) is the projection on the tangent plane at the old estimate \( A_n^0(t + dt/2) \).

We then repeat the procedure starting again from step (2) and iterate as often as necessary in order to bring the error in the inversion \( E_n \) below the required precision (typically around \( 10^{-12} \)). The procedure is repeated for all sites, and at the end of a sweep from 1 to \( L \), one completes an elementary evolution step of \( dt \). For more details about the other aspects of the algorithm and possible improvement using higher-order Ruge–Kutta integration schemes, we refer the reader to the literature on the subject [33, 35, 40].

### Appendix B: Linear spin-wave theory for long-range models

To gain some analytical understanding of the dynamics of the long-range system described by Hamiltonian (2), we employ a linear spin-wave theory (LSWT). This theory is well known to yield good qualitative results in phases with strong magnetic order [52], such as the strongly \( z \)-polarized phase occurring for small \( \theta \) [34]. In our numerical analysis, we will therefore focus on that case, although we will keep our derivations general. An advantage of LSWT is that the long-range interactions are implemented into the formalism without additional complications, as we will sketch now.

#### 1. Determining the ground state

As a first step to finding the ground state of spin waves, it is convenient to rotate the spins into a local, twisted coordinate system \( x', y', z' \), so that the new \( z' \) axis is aligned with the quantization axis. In the antiferromagnetic case of \( \theta > 0 \), a convenient form is to rotate the spin 1/2 operators into \( S'_i = R_i S_i \), where

\[
R_i = \begin{pmatrix}
(-1)^i \cos \gamma & 0 & -\sin \gamma \\
0 & (-1)^{i+1} & 0 \\
(-1)^{i+1} \sin \gamma & 0 & -\cos \gamma
\end{pmatrix}.
\]

Since only \( S^x \) and \( S^z \) operators occur in Hamiltonian (2), it is sufficient to restrict the rotation to the \( xy \) plane. We keep the angle \( \gamma \) free at this stage and will find it later through the minimum of the energy.

In terms of the rotated spin operators, the system Hamiltonian reads

\[
H = 4 \sin \theta \sum_{(ij)} \frac{1}{|i-j|} \left[ (-1)^{i+j} \cos^2 \gamma S_i^x S_j^x + (-1)^{i+j+1} \sin \gamma \cos \gamma (S_i^y S_j^y + S_i^z S_j^z) + (-1)^{i+j} \sin^2 \gamma S_i^z S_j^z \right] - 2 \cos \theta \sum_i (\sin \gamma S_i^x + \cos \gamma S_i^y).
\]

For a state that is strongly polarized along the \( z' \) axis, one can approximate spin-\( S \) operators (here \( S = 1/2 \)) by bosonic operators via the Holstein–Primakoff transformation [52], \( S_i^{z'} \to S - a_i^\dagger a_i \), \( S^+ \to \sqrt{2S} a_i^\dagger \sqrt{1 - \frac{a_i^\dagger a_i}{2S}} \), and

\[
S^- \to \sqrt{2S} \sqrt{1 - \frac{a_i^\dagger a_i}{2S}} a_i. \]

We now insert these into Hamiltonian (A2), neglect contributions beyond linear order in \( a_i^\dagger a_i \), and use that terms that are linear in the boson operators vanish in the minimum of the free energy. More-
over, we apply a Fourier transform \( a_i^\dagger = \frac{1}{\sqrt{L}} \sum_k e^{ikr_i} a_k^\dagger \), leading finally to

\[
H = \sum_k \left[ a_k^\dagger a_k 2 (2 \sin \theta \cos^2 \gamma \gamma_k^{(\alpha)} + \cos \theta \cos \gamma - 4 S \sin \theta \sin^2 \gamma \gamma_k^{(0)} + (a_k^\dagger a_{-k} + a_k a_{-k}) 2 S \sin \theta \cos^2 \gamma \gamma_k^{(\alpha)} \right] + 2 S \sin \theta \cos^2 \gamma \sum_k \gamma_k^{(\alpha)} + L (2 S)^2 \sin \theta \sin^2 \gamma \gamma_k^{(0)} - 2 L S \sin \gamma \cos \gamma ,
\]

where we defined

\[
\gamma_k^{(\alpha)} = \sum_{\delta > 0} (-1)\delta^\alpha \cos k \delta .
\]

This last abbreviation incorporates the entire long-range nature of the system, preserving the extreme simplicity and elegance of LSWT.

Hamiltonian (A3) can now be diagonalized as usual by a Bogolioubov transformation, \( a_k = \cosh \beta_k \alpha_k + \sinh \beta_k \alpha_k^\dagger, \quad a_k^\dagger = \sinh \beta_k \alpha_k + \cosh \beta_k \alpha_k^\dagger \). Demanding that the \( \alpha_k \) obey bosonic commutation relations, and that only terms proportional to \( \alpha_k^\dagger \alpha_k \) yield a contribution to \( H \), one obtains the Bogolioubov angles \( \cosh 2\beta_k = B_k/\omega_k \), \( \sinh 2\beta_k = -2A_k/\omega_k \), and

\[
H = \sum_k \omega_k \left( \alpha_k^\dagger \alpha_k + \frac{1}{2} \right) + \sum_k \left( A_k - \frac{1}{2} B_k - 2 S \cos \theta \cos \gamma + (2 S)^2 \sin \theta \sin^2 \gamma \gamma_k^{(0)} \right) ,
\]

where

\[
B_k = 4 S \sin \theta \cos^2 \gamma \gamma_k^{(\alpha)} + 2 \cos \theta \cos \gamma - 8 S \sin \theta \sin^2 \gamma \gamma_k^{(0)} , \quad A_k = 2 S \sin \theta \cos^2 \gamma \gamma_k^{(\alpha)} ,
\]

and with dispersion relation

\[
\omega_k = \sqrt{B_k^2 - 4 A_k^2}.
\]

The ground state of Hamiltonian (A5), \( |\psi_{\text{GS}}\rangle \), is found as the vacuum of Bogolioubov particles, \( \alpha_k |0\rangle = 0 \forall k \). We can now determine the free-energy minimum \( \gamma \) by minimizing \( \langle \psi_{\text{GS}} | H | \psi_{\text{GS}} \rangle \). Alternatively, one can demand that terms that are linear in the spin-wave operators vanish at the energy minimum, which gives a condition on \( \gamma \) as a function of \( \theta \). For the case of \( \theta = \pi/20 \) that we use in the main text, we find \( \gamma = 0 \) independent of \( \alpha \), and the spins are strongly polarized in negative \( z \) direction.

### 2. Spin-wave group velocity

The dispersion relation (A7) determines the group velocity, which is of the form

\[
v_g \equiv \frac{\partial \omega_k}{\partial k} = \frac{c_1 \partial \gamma_k^{(\alpha)}}{\sqrt{c_2 \gamma_k^{(\alpha)} + c_3}},
\]

with \( c_{1,2,3} \) constants. Divergences of \( v_g \) (as given in the main text, see Fig. 2) can hence be found easily by analyzing \( \gamma_k^{(\alpha)} \) and

\[
\frac{\partial \gamma_k^{(\alpha)}}{\partial k} = -\sum_{\delta > 0} (-1)^\delta \delta^\alpha \sin k \delta .
\]

As illustrated in the main text for the example of \( \theta = \pi/20 \) (Fig. 2b), there are two transitions that can be found generically through an analysis of \( v_{\text{max}} = \max_k v_g(k) \) as a function of \( \alpha \). For \( \alpha > 2 \), \( v_{\text{max}} \) is almost constant as a function of \( \alpha \), whereas below it, it rises rather steeply with decreasing \( \alpha \). This indicates a transition in the dynamical behavior of the system. Additionally, for \( \alpha > 2 \) the \( k \)-value where \( v_{\text{max}} \) is achieved lies around \( k = \pi/2 \) and changes slowly with decreasing \( \alpha \), whereas at \( \alpha = 2 \), it transitions to \( k = \pi \) (Fig. A1, left panel).

In the range \( 1 < \alpha < 2 \), although \( v_{\text{max}} \) achieves large values, \( v_{\text{max}}/L \) scales to zero with increasing system size, indicating the locality of information in these systems. This changes drastically at \( \alpha \leq 1 \), where \( v_{\text{max}}/L \) increases with system size (see discussion in the main text). In that regime, the fastest mode reaches the boundaries earlier for larger systems; the information is distributed essentially instantaneously over the entire chain. Actually, the number of modes diverges for which the time to reach the boundary decreases with system size. Conse-
To evaluate the time evolution under $H$, we make use of the fact that all involved states remain Gaussian at all times. Since Hamiltonian (A3) is quadratic in the boson operators, its ground state is completely determined by the correlators

$$F_{ij} \equiv \langle \psi_{GS} | a_i^\dagger a_j | \psi_{GS} \rangle = \frac{1}{2} \delta_{ij} \frac{1}{2L} \sum_{k} \frac{B_k}{\omega_k} e^{ik(r_j-r_i)},$$

(A10a)

$$G_{ij} = \langle \psi_{GS} | a_i^\dagger a_j^\dagger | \psi_{GS} \rangle = \frac{1}{2L} \sum_{k} -2A_k \frac{1}{\omega_k} e^{ik(r_j-r_i)}$$

(A10b)

At time $t = 0$, we quench the system with a spin flip at site $n_0$, corresponding in LSWT to the operator $S^z_n \equiv \frac{1}{2}(a_m - a_m^\dagger)$, and the state becomes $|\psi_0\rangle = S^z_{n_0} |\psi_{GS}\rangle / \sqrt{N}$, where $N = \langle \psi_{GS} | S^z_{n_0} a_m a_m^\dagger | \psi_{GS} \rangle + \frac{1}{2}$ is the normalization [53]. Since the initial state is Gaussian, expectation values after the quench such as $\langle a_i^\dagger a_j \rangle (t = 0) \equiv \langle \psi_{GS} | S^z_{n_0} a_i^\dagger a_j a_m a_m^\dagger | \psi_{GS} \rangle / N$ can be decomposed into a combination of expectation values of two-point correlations before the quench, using Wick’s theorem [53]. For example, $\langle \psi_{GS} | a^\dagger_i a^\dagger_j a_i a_j | \psi_{GS} \rangle = \langle \psi_{GS} | a^\dagger_i a^\dagger_j | \psi_{GS} \rangle \langle a_j a_i | \psi_{GS} \rangle + \langle \psi_{GS} | a^\dagger_i a^\dagger_j a_i | \psi_{GS} \rangle + \langle \psi_{GS} | a^\dagger_i a^\dagger_j a_j | \psi_{GS} \rangle$. Since the correlations after the quench remain Gaussian, and since a Gaussian state remains Gaussian under the application of a quadratic Hamiltonian, it is sufficient to consider the time evolution of the two-point correlators.

### Appendix C: Power-law interactions are reproducing

if and only if $\alpha > 1$

Typical Lieb–Robinson bounds with the associated velocity are defined for short-range interacting systems, i.e., interactions that decay at least exponentially with distance $i-j$ between lattice sites $i$ and $j$. For interactions
it is assured to converge for decays at least as fast as 

\[ (\frac{3}{2} + m)^\alpha (\frac{3}{2} - m)^\alpha \]

for some constant \( \lambda \). For simplicity, we discuss here open boundary condition for a chain of length \( L + 1 \) with \( L \) even. For a power law \( K(i-j) = 1/|i-j|^\alpha \), this condition is fulfilled if the decay is faster than \( \alpha \geq 1 \) and violated for \( \alpha < 1 \), as we will show now.

1. Power-law interactions are reproducing if \( \alpha > 1 \)

It is convenient to rewrite the condition (A1) using the definition

\[ P(i,j) \equiv \sum_{m=-\frac{L}{2}, m \neq i,j} \frac{|i-j|^\alpha}{|i-m|^\alpha |m-j|^\alpha}, \quad (A2) \]

so that it becomes \( P(i,j) \leq \lambda \). To show that \( 1/|i-j|^\alpha \) is reproducing for \( \alpha \geq 1 \), we need to demonstrate that \( P(i,j) \) converges with \( \alpha \geq 1 \). The only way it can increase is by increasing \( \delta \) evens and \( \delta \) even increasing. Then,

\[
P \left( -\frac{\delta}{2}, \frac{\delta}{2} \right) = \sum_{|m| > \frac{L}{2}} \frac{\delta^\alpha}{(m + \frac{L}{2})^\alpha (m - \frac{L}{2})^\alpha} + \sum_{|m| < \frac{L}{2}} \frac{\delta^\alpha}{(\frac{3}{2} + m)^\alpha (\frac{3}{2} - m)^\alpha}. \quad (A3)
\]

Let us treat the two sums separately. The first sum is upper bounded by \( \sum_{|m| > \frac{L}{2}} \frac{\delta^\alpha}{(m + \frac{L}{2})^\alpha (m - \frac{L}{2})^\alpha} \), the last term of which reads \( M \equiv \frac{\delta^\alpha}{(L - \delta)^2} \). From this, it would seem that this sum converges for \( \alpha > 1/2 \). However, one can consider a more demanding scenario, which shows that convergence is reached only for \( \alpha > 1 \), namely, if one lets \( \delta \) increase with system size, \( \delta = \delta(L) = cL^\beta \). Here, the condition \( \delta < L \) demands \( \beta \leq 1 \). Then,

\[
M = \frac{4^\alpha c^\alpha L^{\alpha \beta}}{L^{2\alpha}(1 - cL^{\beta - 1})^\alpha} \leq \frac{4^\alpha c^\alpha}{L^\alpha(1 - \epsilon)^\alpha}, \quad (A4)
\]

where we used \( \beta \leq 1 \) to bound \( L^{\alpha \beta} \leq L^\alpha \) and \( cL^{\beta - 1} \leq \epsilon \) (with \( \epsilon \) arbitrarily small for \( \beta < 1 \), provided \( L \) is sufficiently large, and \( \epsilon = c < 1 \) for \( \beta = 1 \)). Therefore, \( M \) decays at least as fast as \( L^{-\alpha} \), meaning that the sum over it is assured to converge for \( \alpha > 1 \).

For constant \( \delta \), the second sum in Eq. (A3) is constant. The only way it can increase is by increasing \( \delta \) in some way with \( L \). To study if this can make it diverge, consider the difference of when it is evaluated at \( \delta \) and \( \delta + 2 \),

\[
\sum_{m=\delta+1}^{L/2+1} \frac{(\delta + 2)^\alpha}{(\frac{3}{2} + m + 1)^\alpha (\frac{3}{2} + m)^\alpha} - \sum_{m=\delta}^{L/2} \frac{\delta^\alpha}{(\frac{3}{2} + m)^\alpha (\frac{3}{2} - m)^\alpha} = 2 \sum_{0 < m < \frac{L}{2}} \frac{1}{(\frac{3}{2} - m)^\alpha} \left[ \frac{(\delta + 2)^\alpha}{(\frac{3}{2} + m + 2)^\alpha} - \frac{\delta^\alpha}{(\frac{3}{2} + m)^\alpha} \right] + \frac{(\delta + 2)^\alpha}{(\frac{3}{2} + 2L)^\alpha} - \frac{\delta^\alpha}{(\frac{3}{2})^\alpha}, \quad (A5)
\]

where we regrouped the terms of the two sums into contributions from the outermost summands, the one at the origin, and the additional summands that are inserted beside the origin upon increasing \( \delta \). One can show for the first sum of this expression that all terms are negative, as is the case for the contribution at the origin. Now, we only have to show that the last few terms decay sufficiently fast. In fact, they decay as \( \delta^{-\alpha} \), so that even increasing \( \delta \) proportional to \( L \) leads to a convergent sum as long as \( \alpha > 1 \). Therefore, in this case, also the second sum in Eq. (A3) converges. The argumentation here can be carried over to positions deviating from the symmetric case \( i, j = \pm\delta/2 \). We have thus demonstrated that \( K(i-j) \propto 1/|i-j|^\alpha \) is reproducing for \( \alpha > 1 \).

2. Power-law interactions are non-reproducing if \( \alpha \leq 1 \)

To show that \( K(i-j) \) is non-reproducing for \( \alpha \leq 1 \), it is sufficient to demonstrate the divergence of \( P(i,j) \) for a specific case, which can easily be done for the choice \( i = -L/2, j = L/2 \),

\[
P(-L/2, L/2) = \sum_{m=-L/2+1}^{L/2-1} \frac{L^\alpha}{(m + L/2)^\alpha (m - L/2)^\alpha} = \sum_{m=1}^{L-1} \frac{1}{m^\alpha (1 - \frac{m}{L})^\alpha} \geq \sum_{m=1}^{L-1} \frac{1}{m^\alpha} \quad (A6)
\]

The last sum converges towards the Riemann–zeta function \( \zeta(\alpha) \). This lower bound for \( K(i-j) \), therefore, diverges for \( \alpha \leq 1 \), where \( K(i-j) \) is hence non-reproducing. This property explains the violation of the Lieb–Robinson bounds for small \( \alpha \).
unity, which are un-physical in the initial spin Hamiltonian. However, under the time evolution, the density spreads very fast far below 1, so that the mapping between spin-waves and bosons is restored. A possibility to avoid this problem would be to apply only a partial spin flip.