Dark Entangled States by Coherent Absorption in Cascaded Quantum Networks

K. Stannigel[1,2], P. Rabl[1], and P. Zoller[1,2]
[1] *Institute for Quantum Optics and Quantum Information, Austrian Academy of Sciences, 6020 Innsbruck, Austria and*
[2] *Institute for Theoretical Physics, University of Innsbruck, 6020 Innsbruck, Austria*
(Dated: March 16, 2013)

We study the dissipative dynamics and the formation of entangled states in driven cascaded quantum networks, where multiple systems are coupled to a common unidirectional bath. Specifically, we identify the conditions under which emission and coherent reabsorption of radiation drives the whole network into a pure stationary state with non-trivial quantum correlations between the individual nodes. For the case of cascaded two-level systems, we present an explicit preparation scheme that allows for tuning the network into different classes of multi-partite entangled states. We discuss potential realizations of such cascaded networks with optical and microwave photons, where these effects could be used, for example, for new long-distance entanglement distribution protocols.

PACS numbers: 03.67.Bg, 03.65.Yz, 42.50.Lc

Recently, there has been significant interest in controlling the dynamics of open many body quantum systems by engineered couplings to an environment [1]. Various scenarios involving dissipative dynamics for quantum computing, quantum communication and quantum simulation have been described theoretically [2–8], and first experiments demonstrating the dissipative preparation of GHZ states in systems of trapped ions [9] and EPR entangled states of two atomic ensembles [10] have been reported. In these experiments the underlying principle has been to carefully design and implement a many-particle master equation, where a fully dissipative dynamics drives the system into a unique *pure* steady state representing the entangled state of interest. Below we study entanglement formation in a driven many-particle *cascaded quantum network* [11], where the unconventional coupling of multiple systems to a common unidirectional bath offers remarkable new opportunities for a dissipative preparation of highly correlated states.

As illustrated in Fig. 1(a) a cascaded quantum network represents the coupling of N systems to a 1D reservoir, with the unique feature that excitations in the bath can only propagate along a *single* direction, thereby driving successive systems in the network in a unidirectional way. While such a scenario is reminiscent of edge modes in quantum Hall systems [12, 13], various artificial and more controlled realizations based on (integrated) non-reciprocal devices for optical [14, 15] and microwave [16, 17] photons are currently developed. Our goal below is to study the general properties of a driven N-body cascaded network, and to identify conditions for the existence of (pure) entangled states between the individual nodes, which are established under continuous driving due to emission and coherent reabsorption of photons. We will identify these entangled states as *dark states* of the whole cascaded network [18], representing many particle quantum states which are completely decoupled from the bath. In this situation (c.f. Fig. 1(a)) all photons generated in the initial stages of the network are eventually completely (coherently) reabsorbed by the final nodes with no scattered photons emerging from the system. In other words, steady state entanglement is established when the system acts as its own *coherent quantum absorber*. In a more general context, the cascaded network realizes a novel type of non-equilibrium many body system, which by changing the system parameters can be tuned between "dark or passive phases" (with no scattered photons emerging) and "bright or active phases" (with light scattered), while the nodes, represented e.g. by spins, are driven into pure entangled or mixed states, respectively.

Model. We consider the general setting for a cascaded quantum network shown in Fig. 1(a). Here, $N \geq 2$ subsystems located at positions x_i are coupled to a 1D continuum of right-propagating bosonic modes b_ω which we refer to as photons. The whole network can be modeled

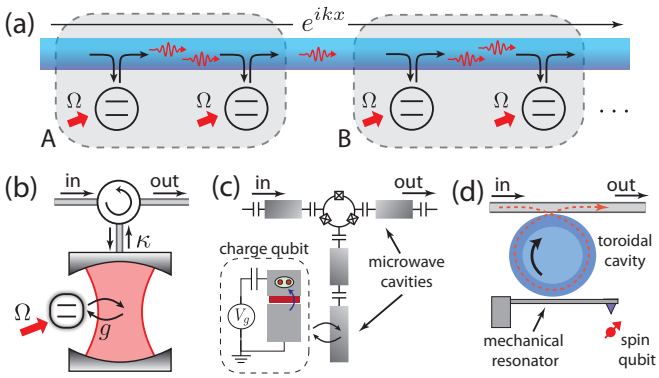

FIG. 1. (color online) (a) Driven cascaded quantum network realized by two-level systems coupled to a unidirectional bath. (b-d) Possible realizations for a single node (see text): (b) generic setup where the uni-directional coupling is achieved by a cavity connected to a circulator, (c) non-reciprocal superconducting circuit based on the proposal of Ref. [16], (d) optomechanical transducer based on toroidal cavities [19].

by a Hamiltonian ($\hbar = 1$)

$$H = \sum_i H_i + H_\mathrm{b} + \sum_i \int g_\omega \left(c_i b_\omega^\dagger e^{i\frac{\omega}{c}x_i} + \mathrm{H.c.} \right) \mathrm{d}\omega, \quad (1)$$

where $H_\mathrm{b} = \int \mathrm{d}\omega\, \omega b_\omega^\dagger b_\omega$ and the integrals run over a broad bandwidth around the characteristic system frequency ω_0. In Eq. (1) the H_i describe the driven dynamics of the individual systems and their coupling to the bath is determined by the 'jump operators' c_i and coupling constants g_ω. Potential realizations of such a setting with two-level systems coupled to photonic waveguides are indicated in Fig. 1(b)-(d) and will be discussed further below.

The system–bath interaction in Eq. (1) breaks time reversal symmetry and while photons can be emitted to the right, drive successive subsystems and eventually leave the network, the reverse processes cannot occur. To study the effects of this unconventional coupling, we eliminate the photons in a Born-Markov approximation and derive a generalized cascaded master equation (ME) for the reduced system density operator ρ [11, 20],

$$\dot{\rho} = \sum_i \mathcal{L}_i \rho - \gamma \sum_{j>i} \left([c_j^\dagger, c_i \rho] + [\rho c_i^\dagger, c_j] \right). \quad (2)$$

Here the first part describes the uncoupled evolution of each subsystem $\mathcal{L}_i \rho = -i[H_i, \rho] + \gamma \mathcal{D}[c_i]\rho$, where the Lindblad terms $\mathcal{D}[x]\rho = x\rho x^\dagger - \{x^\dagger x, \rho\}/2$ describe dissipation due to emission of photons with a rate $\gamma = \pi g_{\omega_0}^2$. The unidirectionality of the bath is reflected by the last term in Eq. (2), which accounts for the possibility to reabsorb photons emitted at system i by all successive nodes located at $x_j > x_i$. The Lindblad form of the ME (2) reads

$$\dot{\rho} = -i[H_\mathrm{casc}, \rho] + \gamma \mathcal{D}[c]\rho, \quad (3)$$

where $H_\mathrm{casc} = \sum_i H_i - i\frac{\gamma}{2} \sum_{j>i} (c_j^\dagger c_i - c_i^\dagger c_j)$ now includes the non-local coherent part of the environment-mediated coupling, while the only decay channel with collective jump operator $c = \sum_i c_i$ is associated with a photon leaving the system to the right [21].

Perfect quantum absorbers. In the following we are interested in steady state situations where every photon emitted within the system is perfectly reabsorbed by successive nodes in the network, such that there is no spontaneous emission via the waveguide output and the system relaxes into a pure steady state $\rho_0 = |\psi_0\rangle\langle\psi_0|$. To identify the general conditions for the existence of such states, we partition the network into two subsystems A and B as indicated in Fig. 1(a), with local Hamiltonians H_A and H_B, and jump operators c_A and c_B, respectively [22]. Then, in Eq. (3), $H_\mathrm{casc} = H_A + H_B - i\frac{\gamma}{2}(c_A c_B^\dagger - c_A^\dagger c_B)$ and $c = c_A + c_B$ and the conditions for the existence of a pure stationary state are (see Ref. [1] and App. A):

$$(\mathbf{I})\ \ (c_A + c_B)|\psi_0\rangle = 0, \qquad (\mathbf{II})\ \ [H_\mathrm{casc}, \rho_0] = 0. \quad (4)$$

The first condition implies that the waveguide output is dark, i.e., $\langle c^\dagger c \rangle = 0$, and the second one ensures stationarity. To exclude trivial cases where $[H_{A,B}, \rho_0] = 0$ and $c_{A,B}|\psi_0\rangle = 0$ are fulfilled for each subsystem separately, we will in addition demand non-vanishing cross-correlations, $\mathcal{C} = \langle c_A^\dagger c_B + c_A c_B^\dagger \rangle - 2\mathrm{Re}\{\langle c_A^\dagger \rangle \langle c_B \rangle\}$. In view of (**I**) this third requirement can be expressed as

$$(\mathbf{III})\ \ \mathcal{C} = -2(\langle c_A^\dagger c_A \rangle - |\langle c_A \rangle|^2) \neq 0, \quad (5)$$

and directly connects the correlations between A and B with the amount of radiation emitted from the first subsystem. Note that for a pure state, $\mathcal{C} \neq 0$ implies that A and B are entangled.

The conditions (**I**) – (**III**) will not be satisfied in general. However, given a system A described by a Hamiltonian H_A and jump operator c_A we can construct a perfect coherent absorber system B as follows. First, we point out that due to the unidirectional coupling the dynamics of A is unaffected by B, and we write a unique stationary solution of $\mathcal{L}_A \rho_A^0 = 0$ in terms of its spectral decomposition, $\rho_A^0 = \sum_k p_k |k\rangle\langle k|$. A pure state of the whole system is then given by $|\psi_0\rangle = \sum_k \sqrt{p_k}|k\rangle_A \otimes |\tilde{k}\rangle_B$, where $|\tilde{k}\rangle = V|k\rangle$ for an arbitrary unitary operation V on subsystem B, which we assume to have the same dimension as A. Now, (**I**) and (**II**) can be satisfied by the choice (see App. A)

$$c_B = -\sum_{n,m} \sqrt{\frac{p_n}{p_m}} \langle m|c_A|n\rangle |\tilde{n}\rangle\langle\tilde{m}|_B, \quad (6)$$

$$H_B = -\frac{1}{2} \sum_{n,m} \left(\sqrt{\frac{p_n}{p_m}} A_{mn} + \sqrt{\frac{p_m}{p_n}} A_{nm}^* \right) |\tilde{n}\rangle\langle\tilde{m}|_B, \quad (7)$$

where $A_{mn} = \langle m|H_{A,\mathrm{eff}}|n\rangle$ and $H_{A,\mathrm{eff}} = H_A - i\frac{\gamma}{2} c_A^\dagger c_A$. While Eq. (6) and (7) define a general absorber system B, we find that for many systems of interest the stationary state ρ_A^0 satisfies $\sqrt{p_n}\langle k|c_A|n\rangle = \sqrt{p_k}\langle n|c_A|k\rangle$ and $\sqrt{p_n}\langle k|H_{A,\mathrm{eff}}|n\rangle = \sqrt{p_k}\langle n|H_{A,\mathrm{eff}}|k\rangle$. In this case, the above relations simplify to $c_B = -V c_A V^\dagger$, $H_B = -V H_A V^\dagger$ and up to unitary basis transformations the absorber system is just the negative counterpart of A. In particular, this situation applies to the examples presented in the following, but also to other relevant cases, such as cascaded Kerr-nonlinear cavities.

Cascaded spin networks. Let us now be more specific and consider a set of N driven two-levels systems ('spins') coupled to a unidirectional bosonic bath as shown in Fig. 1(a). The collective jump operator is $c = \sum_i \sigma_-^i$ and the cascaded Hamiltonian in the frame rotating at the drive frequency reads

$$H_\mathrm{casc} = \sum_i \left(\frac{\delta_i}{2} \sigma_z^i + \Omega_i \sigma_x^i \right) - i\frac{\gamma}{2} \sum_{j>i} \left(\sigma_+^j \sigma_-^i - \sigma_-^j \sigma_+^i \right), \quad (8)$$

where the σ_k are the usual Pauli operators, the Ω_i are local Rabi frequencies, and the δ_i are the detunings of the spins from the driving field.

For $N = 2$ the dark state condition **(I)** restricts $|\psi_0\rangle$ to the subspace spanned by $|gg\rangle$ and the singlet $|S\rangle = (|eg\rangle - |ge\rangle)/\sqrt{2}$. Condition **(II)** can then be satisfied for $\Omega_1 = \Omega_2 \equiv \Omega$ and any $\delta_1 = -\delta_2 \equiv \delta$, for which we obtain the unique and pure steady state $|\psi_0\rangle = |\mathcal{S}_\delta\rangle$, where

$$|\mathcal{S}_\delta\rangle = \frac{1}{\sqrt{1+|\alpha|^2}}\left(|gg\rangle + \alpha|S\rangle\right), \qquad \alpha = \frac{2\sqrt{2}\Omega}{i\gamma - 2\delta}. \quad (9)$$

The two spins thus realize a source and a matched absorber in the sense introduced above, and for the matrix representations of the various operators we can identify $c_A = \sigma_-$, $c_B = -Vc_AV^\dagger = \sigma_-$ and $H_B = -VH_AV^\dagger$, using $V = \sigma_z$. For strong driving, $|\alpha| \gg 1$, the state $|\mathcal{S}_\delta\rangle$ approaches the singlet $|S\rangle$, where also $\mathcal{C} \to -1$.

While for larger N a direct search for possible dark states is hindered by the exponential growth of the subspace defined by $c|\psi_0\rangle = 0$, we can use the state (9) as a starting point and solve the cascaded system iteratively 'from left to right': Suppose that for $\Omega_i = \Omega$ and $\delta_1 = -\delta_2$ the first two spins have evolved into the dark state $|\mathcal{S}_{\delta_1}\rangle$ such that no more photons are emitted into the waveguide. Then, the following two spins will evolve into the dark state $|\mathcal{S}_{\delta_3}\rangle$, provided that $\delta_3 = -\delta_4$. By iterating this argument we see that for any detuning profile with $\delta_{2i-1} = -\delta_{2i}$ $(i = 1, 2, \ldots)$ the steady state of ME (3) is given by (see App. B)

$$|\mathcal{S}^0\rangle = |\mathcal{S}_{\delta_1}\rangle_{12} \otimes |\mathcal{S}_{\delta_3}\rangle_{34} \otimes \ldots . \quad (10)$$

In particular, for $\delta_i \simeq 0$, a strongly driven cascaded spin system relaxes into a chain of pairwise singlets.

The dimer structure of the state $|\mathcal{S}^0\rangle$ reflects that radiation emitted from one node is immediately reabsorbed by the following one. We now consider more general situations, where this reabsorption can also occur by several of the following spins, leading to stationary states with multi-partite correlations. To this end, we note that by starting from Eq. (10) we can construct another dark state $|\mathcal{S}'\rangle = U|\mathcal{S}^0\rangle$ by any global unitary operation U which commutes with the collective jump operator c, while implementing the Hamiltonian $H' = UH_{\rm casc}U^\dagger$ would insure stationarity. However, H' would generally contain additional non-local terms, and to avoid this we must restrict ourselves to unitaries U under which $H_{\rm casc}$ is *form invariant*. As an example, we write $H_{\rm casc} \equiv H_{\rm casc}(\Delta)$, where $\Delta = (\delta_1, \delta_2, ...)$ is the detuning profile, and introduce the nearest-neighbor operations

$$U_i(\theta_i) = \exp\left[i\frac{\theta_i}{4}\left(\vec{\sigma}_i + \vec{\sigma}_{i+1}\right)^2\right]. \quad (11)$$

Then, by choosing $\tan(\theta_i) = (\delta_{i+1} - \delta_i)/\gamma$ we obtain

$$H' = U_i(\theta_i)\, H_{\rm casc}(\Delta)\, U_i^\dagger(\theta_i) = H_{\rm casc}(\Delta'), \quad (12)$$

with a new detuning profile $\Delta' = P_{i,i+1}\Delta$, where $P_{i,i+1}$ denotes the permutation of δ_i and δ_{i+1} (see App. B).

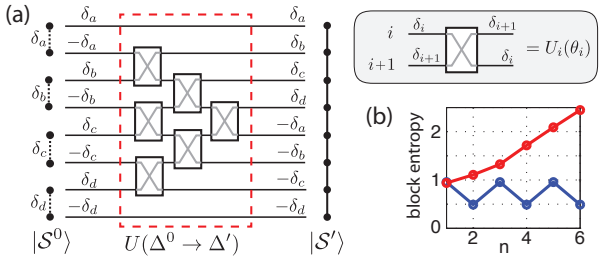

FIG. 2. (color online) (a) Circuit model for constructing the state $|\mathcal{S}'\rangle$ in Eq. (13) for the example $\Delta' = (\delta_a, \delta_b, \ldots, -\delta_a, -\delta_b, \ldots)$. Each line represents a spin and is labeled by its detuning. A box connecting two lines denotes a unitary operation $U_i(\theta_i)$ and the corresponding exchange of detunings δ_i and δ_{i+1} (see text). (b) Scaling of the von-Neumann entropy of the first n spins in a network of $N = 12$ nodes for $\Omega = 2\gamma$. Upper curve: detuning profile as in (a) with $|\delta_i| = \gamma/3$. Lower curve: state corresponding to the detuning profile $\Delta' = (0, \gamma, -\gamma, \gamma, \ldots, -\gamma, 0)/3$.

Thus, by starting from a set Δ^0 of alternating detunings as defined before Eq. (10), we can simply swap the detunings of nodes i and $i+1$ to implement a new cascades spin network with a unique stationary state $|\mathcal{S}'\rangle = U_i(\theta_i)|\mathcal{S}^0\rangle$. By repeating this argument, we obtain a different pure steady state for each permutation Δ' of Δ^0. This class of states is given by

$$|\mathcal{S}'\rangle = U(\Delta^0 \to \Delta')|\mathcal{S}^0\rangle, \quad (13)$$

where $U(\Delta^0 \to \Delta')$ is a product of nearest neighbor operations $U_i(\theta_i)$, specified by the sequence of permutations required for transforming Δ^0 into Δ'. A graphical representation of $U(\Delta^0 \to \Delta')$ in terms of a circuit model is shown in Fig. 2(a).

Discussion. For large detuning differences $|\delta_i - \delta_{i+1}| \gg \gamma$ the transformations (11) are SWAP operations between neighboring sites and in this limit the states $|\mathcal{S}'\rangle$ remain approximately two-partite entangled, but with singlets shared between arbitrary nodes in the network. In contrast, for $|\delta_i - \delta_{i+1}| \approx \gamma$ the U_i correspond to highly entangling $\sqrt{\rm SWAP}$ operations. Then, the entanglement structure can be much richer and in general the states $|\mathcal{S}'\rangle$ contain multi-partite entanglement between several or even all nodes. While in this case a full characterization is difficult, we point out that the U_i conserve total angular momentum such that the $|\mathcal{S}'\rangle$ approach multi-spin singlets in the strong driving limit. The amount of entanglement between subsystems now depends very much on the choice of the detuning profile, as can be seen from the two examples displayed in Fig. 2(c), showing oscillating and linearly growing block entropy, respectively. More generally, we see that the cascaded network can be driven into different types of two- and multi-partite entangled states by simply adjusting local detunings. This is illustrated in Fig. 3, where an adiabatic variation of the detunings in a six-node network is used to prepare pure

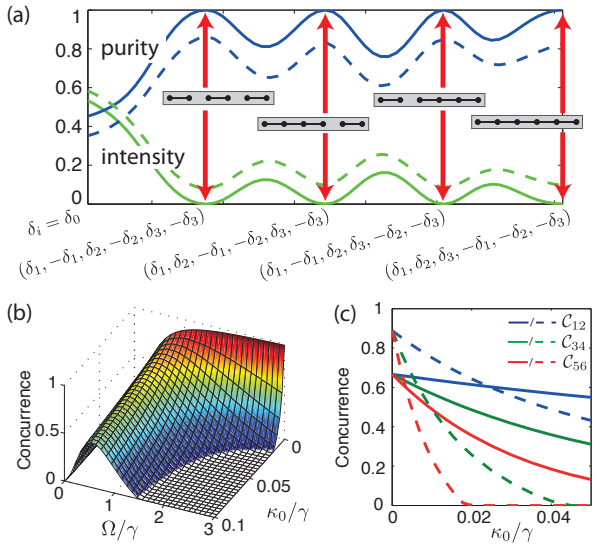

FIG. 3. (color online). (a) Purity and output intensity $\langle c^\dagger c\rangle$ in a six-spin network, where detunings δ_i are interpolated linearly between the profiles given on the horizontal axis. Red arrows indicate the entanglement structure of the dark states. The solid lines show the ideal case ($\kappa_0 = 0$) and the dashed lines include a finite on-site decay $\kappa_0 = 0.0025\gamma$. The other parameters are $\Omega = \gamma$, $\delta_0 = 0.2\gamma$, $\delta_1 = \gamma$, $\delta_2 = \gamma/2$, $\delta_3 = 0$. (b,c) Influence of onsite decays on (b) steady state concurrence for $N = 2$ spins and (c) concurrences $\mathcal{C}_{i,i+1}$ of reduced two-spin density matrices $\rho_{i,i+1}$ for $N = 6$ spins. In both plots $\delta_i = 0$ and in (c) $\Omega/\gamma = 1/2$ (solid lines) and $\Omega/\gamma = 1$ (dashed lines).

steady states with 2-, 4- and 6-partite entanglement, separated by 'bright' (mixed state) phases where the conditions in Eq. (4) are violated. Numerical calculations for small system suggest that the preparation time of these steady states scales efficiently with N.

Imperfections. Under realistic conditions various imperfections like onsite decays or losses in the waveguide can violate the exact dark state condition and the system then evolves to a mixed ('bright') steady state (see dashed lines Fig. 3(a)). As an example, we add local decays $\mathcal{L}'\rho = \kappa_0 \sum_i \mathcal{D}[\sigma_-^i]\rho$ to our ME, and Fig. 3(b) and (c) show the resulting steady state entanglement in the presence of such losses. For $N = 2$ entanglement is quite robust and optimized for intermediate driving strengths Ω. For larger systems, the scattering of photons from the first nodes also affects successive spins, as shown in Fig. 3(c) for the dimer chain $|\mathcal{S}^0\rangle$. However, we observe a tradeoff between the maximal achievable entanglement and the robustness of the state, and Fig. 3(a) also shows that for a fixed N, different bi- and multi-partite entangled states are affected equally. We find that these features are quite generic and similar dependencies are found for various other sources of decoherence.

Implementations. The two key ingredients for realizing a cascaded network as shown in Fig. 1(a) are (i) a strong coupling of single two-level systems to a 1D waveguide, and (ii) the implementation of low loss non-reciprocal devices for directional routing of photons. In principle, requirement (i) can be fulfilled in the optical and microwave domain along the lines of Refs. [23, 24], and for (ii) one can, e.g., use standard circulators based on the Faraday effect. However, the on-going development of non-reciprocal on-chip devices for optical [14, 15] and microwave [16, 17] photons promises alternative, integrated realizations. A generic implementation of a single node, which also loosens requirement (i) is shown in Fig. 1(b), where a two-level system is strongly coupled to a cavity, whose output port is connected to a circulator. In the bad cavity limit $\kappa \gg g$, a series of these nodes gives rise to the desired model (1), and a particular realization in the context of circuit cavity QED [25] is shown in Fig. 1(c). Finally, we note that additional systems like optomechanical transducers (Fig. 1(d)) have been proposed to realize a similar unidirectional coupling [19].

Conclusions. We have shown that photon emission and coherent reabsorption processes in cascaded quantum systems can lead to the formation of pure and highly entangled stationary states. In the case of spin networks, this mechanism provides a tunable dissipative preparation scheme for a whole class of multi-partite entangled states. Apart from potential quantum communication applications [7, 26, 27] our findings show that such driven cascaded networks realize a novel type of non-equilibrium quantum many-body system, which can be realized with currently developed integrated optical systems or superconducting devices.

Acknowledgments. The authors thank B. Kraus for valuable discussions. This work was supported by the EU network AQUTE and the Austrian Science Fund (FWF) through SFB FOQUS and the START grant Y 591-N16.

APPENDIX A: GENERAL CONSTRUCTION OF COHERENT QUANTUM ABSORBERS

Given a system A in terms of its Hamiltonian H_A and jump operator c_A, we seek a suitable system B described by H_B and c_B which perfectly reabsorbs the output field of A in steady state. In the following, we construct system B by requiring that the total cascaded system evolves into a pure steady state. The cascaded ME already given in the main text is

$$\dot{\rho} = -i[H_{\text{casc}}, \rho] + \gamma \mathcal{D}[c]\rho, \tag{14}$$

where $\mathcal{D}[x]\rho = x\rho x^\dagger - \{x^\dagger x, \rho\}/2$ is a Lindblad term,

$$H_{\text{casc}} = H_A + H_B - i\frac{\gamma}{2}(c_A c_B^\dagger - c_A^\dagger c_B) \tag{15}$$

is the total Hamiltonian and $c = c_A + c_B$ the collective jump operator. The above ME is to be understood in a suitable rotating frame to account for classical driving fields. Assuming that c has no eigenvectors, a pure state $|\psi_0\rangle$ is a stationary state of Eq. (14) if and only if [1]

$$c|\psi_0\rangle = 0\,, \qquad [H_{\text{casc}}, |\psi_0\rangle\langle\psi_0|] = 0\,. \tag{16}$$

Due to the cascaded nature of the interaction we can always solve for the reduced steady state ρ_A^0 of system A without knowing anything about B. Assuming ρ_A^0 is unique, it can be determined from $\mathcal{L}_A\rho_A^0 = 0$, where $\mathcal{L}_A\rho = -i[H_A, \rho] + \gamma\mathcal{D}[c_A]\rho$. We write ρ_A^0 in terms of its spectral decomposition

$$\rho_A^0 = \sum_k p_k |k\rangle\langle k|\,, \tag{17}$$

where we assume that the eigenvalues p_k are positive and non-degenerate. A pure steady state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ of the whole system can then be written as a purification of ρ_A^0, i.e., $|\psi_0\rangle = \sum_k \sqrt{p_k}|k\rangle_A \otimes |\tilde{k}\rangle_B$, where $|\tilde{k}\rangle = V|k\rangle$ for an arbitrary unitary operation V on subsystem B. In the following we write out the tensor-products more explicitly, such that the dark-state condition $c|\psi_0\rangle = 0$ reads

$$(c_A \otimes \mathbb{1} + \mathbb{1} \otimes c_B)\,|\psi_0\rangle = 0\,, \tag{18}$$

where

$$c_A = \sum_{n,m} \langle n|c_A|m\rangle |n\rangle\langle m|\,, \quad c_B = \sum_{n,m} \langle\tilde{n}|c_B|\tilde{m}\rangle |\tilde{n}\rangle\langle\tilde{m}|.$$

To proceed, we plug the ansatz for $|\psi_0\rangle$ into the dark state condition in Eq. (18), which yields

$$\sum_{k,n} \sqrt{p_k}\left(\langle n|c_A|k\rangle |n\rangle \otimes |\tilde{k}\rangle + \langle\tilde{n}|c_B|\tilde{k}\rangle |k\rangle \otimes |\tilde{n}\rangle\right) = 0,$$

and after relabeling the indices we obtain

$$\sum_{k,n}\left(\sqrt{p_k}\langle n|c_A|k\rangle + \sqrt{p_n}\langle\tilde{k}|c_B|\tilde{n}\rangle\right)|k\rangle \otimes |\tilde{n}\rangle = 0. \tag{19}$$

Since the $|k\rangle$ form an orthonormal basis, this condition is equivalent to

$$\sqrt{p_n}\langle\tilde{k}|c_B|\tilde{n}\rangle = -\sqrt{p_k}\langle n|c_A|k\rangle, \tag{20}$$

which fixes the operator c_B to be

$$c_B = -\sum_{n,m}\sqrt{\frac{p_n}{p_m}}\langle m|c_A|n\rangle |\tilde{n}\rangle\langle\tilde{m}|. \tag{21}$$

To construct the Hamiltonian H_B for system B we exploit the second condition $[H_{\text{casc}}, \rho_0] = 0$ needed for a pure steady state, which is equivalent to $H_{\text{casc}}|\psi_0\rangle = \lambda|\psi_0\rangle$ for $\lambda \in \mathbb{R}$. Note that $c|\psi_0\rangle = 0$ implies

$$i\frac{\gamma}{2}(c_A \otimes c_B^\dagger - c_A^\dagger \otimes c_B)|\psi_0\rangle = i\frac{\gamma}{2}(c_A^\dagger c_A \otimes \mathbb{1} - \mathbb{1} \otimes c_B^\dagger c_B)|\psi_0\rangle,$$

such that this condition reads

$$\left(H_{A,\text{eff}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{B,\text{eff}}^\dagger\right)|\psi_0\rangle = \lambda|\psi_0\rangle\,, \tag{22}$$

where we have introduced the effective non-hermitian Hamiltonians $H_{j,\text{eff}} = H_j - i\frac{\gamma}{2}c_j^\dagger c_j$. Since a finite λ would only lead to a global shift of H_B below, we can assume $\lambda = 0$ without loss of generality. We write the Hamiltonian as

$$H_B = \sum_{n,m}\langle\tilde{n}|H_B|\tilde{m}\rangle |\tilde{n}\rangle\langle\tilde{m}|, \tag{23}$$

and to determine the matrix elements in the expansion we start from Eq. (22) and proceed as for the dark state condition Eq. (18) with the replacements $c_A \to H_{A,\text{eff}}$ and $c_B \to H_{B,\text{eff}}^\dagger$. As an intermediate result this yields

$$\langle\tilde{n}|H_B|\tilde{m}\rangle = -\sqrt{\frac{p_n}{p_m}}\langle m|H_{A,\text{eff}}|n\rangle - i\frac{\gamma}{2}\langle\tilde{n}|c_B^\dagger c_B|\tilde{m}\rangle. \tag{24}$$

To express the right-hand side of this equation fully in terms of operators on A, we employ the identity

$$\langle\tilde{n}|c_B^\dagger c_B|\tilde{m}\rangle = \sqrt{\frac{1}{p_m p_n}}\langle m|c_A\rho_A^0 c_A^\dagger|n\rangle, \tag{25}$$

which can be derived with the help of Eq. (20), and then make use of the stationarity of system A,

$$\gamma c_A \rho_A^0 c_A^\dagger = i(H_{A,\text{eff}}\rho_A^0 - \rho_A^0 H_{A,\text{eff}}^\dagger). \tag{26}$$

Eq. (24) then becomes

$$\langle\tilde{n}|H_B|\tilde{m}\rangle = -\frac{1}{2}\left(\sqrt{\frac{p_n}{p_m}}\langle m|H_{A,\text{eff}}|n\rangle \right. \tag{27}$$

$$\left. + \sqrt{\frac{p_m}{p_n}}\langle m|H_{A,\text{eff}}^\dagger|n\rangle\right) \tag{28}$$

which determines the Hamiltonian of system B. The expressions (21) and (27) are the results quoted in the main text.

APPENDIX B: DARK STATES OF CASCADED SPIN NETWORKS

Uniqueness of $|\mathcal{S}^0\rangle$

We show by explicit construction that the state $|\mathcal{S}^0\rangle$ given in Eq. (10) is the unique steady state of the ME (3) with $H_{\rm casc}$ defined in Eq. (8) for $\delta_{2i} = -\delta_{2i-1}$ and equal Rabi-frequencies. To do so, we exploit the fact that the cascaded interaction allows for a successive construction of the steady state and start with the case $N = 2$. In this case, the steady state ρ_2^0 is obtained by solving $\mathcal{L}^{(2)}\rho_2^0 = 0$, where $\mathcal{L}^{(2)} = \mathcal{L}_{12}$ is given by the block-wise Liouvillian

$$\mathcal{L}_{i,i+1}\rho = -\,i\left[\frac{\delta_i}{2}(\sigma_i^z - \sigma_{i+1}^z) + \Omega(\sigma_i^x + \sigma_{i+1}^x), \rho\right] \quad (29)$$

$$+\,\frac{\gamma}{2}\mathcal{D}[c_{i,i+1}]\rho \quad (30)$$

with $c_{i,j} = \sigma_i^- + \ldots + \sigma_j^-$. We have already seen in the main text that a solution is given by $\rho_2^0 = |\mathcal{S}_{\delta_1}\rangle\langle\mathcal{S}_{\delta_1}|$, and by calculating the characteristic polynomial of $\mathcal{L}_{12}$ one can show that there is only one zero eigenvalue for $\gamma > 0$, such that this solution is also unique.

We continue with $N = 4$ and write the ansatz for the steady state as $\rho_4^0 = |\mathcal{S}_{\delta_1}\rangle\langle\mathcal{S}_{\delta_1}| \otimes \mu$, where μ is a two-node density matrix. The four-node Liouvillian can be rewritten as

$$\mathcal{L}^{(4)}\rho = \mathcal{L}_{12}\rho + \mathcal{L}_{34}\rho - \gamma\left([c_{34}^\dagger, c_{12}\rho] + [\rho c_{12}^\dagger, c_{34}]\right) \quad (31)$$

and we note that $\mathcal{L}_{12}|\mathcal{S}_{\delta_1}\rangle\langle\mathcal{S}_{\delta_1}| = 0$ as well as $c_{12}|\mathcal{S}_{\delta_1}\rangle = 0$, such that the equation $\mathcal{L}^{(4)}\rho_4^0 = 0$ simplifies to $\mathcal{L}_{34}\mu = 0$. However, this is just the two-node problem we have already solved and the unique solution is thus given by $\rho_4^0 = |\mathcal{S}_{\delta_1}\rangle\langle\mathcal{S}_{\delta_1}|\otimes|\mathcal{S}_{\delta_3}\rangle\langle\mathcal{S}_{\delta_3}|$. By iterating this argument in blocks of two spins, we obtain the steady state of Eq. (10).

Unitary form invariance

We briefly demonstrate that the statement (12) is true by calculating $U_i H_{\rm casc} U_i^\dagger$. To this end, we write $j \equiv i+1$ for brevity and rearrange $H_{\rm casc}$ of Eq. (8) as follows:

$$\begin{aligned} H_{\rm casc} = &\sum_{k\neq i,j}\left(\frac{\delta_k}{2}\sigma_z^k + \Omega\sigma_x^k\right) - i\frac{\gamma}{2}\sum_{k>l}^{k,l\neq i,j}(\sigma_+^k\sigma_-^l - \sigma_-^k\sigma_+^l) \\ &- i\frac{\gamma}{2}\left(c_{i,j}^\dagger c_{1,i-1} + c_{j+1,N}^\dagger c_{i,j} - \text{H.c.}\right) \\ &+ \frac{\delta_i+\delta_j}{2}\frac{1}{2}(\sigma_z^i + \sigma_z^j) + \Omega(\sigma_x^i + \sigma_x^j) \\ &+ \left[\frac{\delta_i-\delta_j}{2}\frac{1}{2}(\sigma_z^i - \sigma_z^j) - i\frac{\gamma}{2}(\sigma_+^j\sigma_-^i - \sigma_-^j\sigma_+^i)\right] \end{aligned}$$

where we have introduced the piecewise jump operator $c_{k,l} = \sigma_k^- + \sigma_{k+1}^- + \ldots + \sigma_l^-$. Note that $U_i(\theta)$ commutes with the first three lines and we thus focus on the last one. We abbreviate the operators appearing there by $A = (\sigma_z^i - \sigma_z^j)/2$ and $B = \sigma_j^+\sigma_i^- - \sigma_j^-\sigma_i^+$ and note that they transform into one another:

$$U_i(\theta)AU_i^\dagger(\theta) = A\cos 2\theta + iB\sin 2\theta \quad (32)$$

$$U_i(\theta)BU_i^\dagger(\theta) = B\cos 2\theta + iA\sin 2\theta \quad (33)$$

Introducing the difference of detunings $\delta = (\delta_i - \delta_j)/2$, a non-trivial form-invariance of the Hamiltonian $H_{\rm casc}$ under $U_i(\theta)$ is realized if

$$U_i(\theta)\left[\delta A - i\frac{\gamma}{2}B\right]U_i^\dagger(\theta) = -\delta A - i\frac{\gamma}{2}B \quad (34)$$

That is, we require the detunings to swap and the cascaded part to remain invariant. It is easy to check that this requirement results in two equations, which are both solved for the choice $\tan\theta = -2\delta/\gamma = (\delta_{i+1} - \delta_i)/\gamma$.

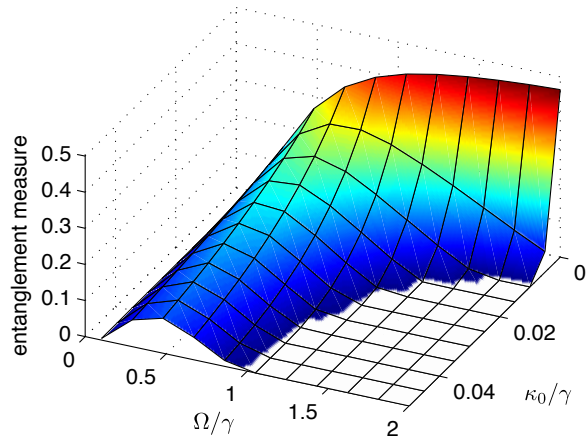

FIG. 4. (color online). Influence of a finite onsite decay rates κ_0 on the steady state four-partite entanglement in a four-node network. The decay is modeled by the additional Lindblad terms $\sum_i \kappa_0\mathcal{D}[\sigma_i^-]$ in the ME and the four-partite entanglement is quantified by the measure of Ref. [28], which is bounded by 1/2 in the present situation. The detuning profile is given by $\Delta' = (0, -\gamma/2, \gamma/2, 0)$.

Multi-partite entanglement in the presence of imperfections

To study the robustness of multi-partite entanglement in the presence of imperfections, we employ the entanglement measure proposed in Ref. [28]. It can be evaluated in a straight-forward way and Fig. 4 displays the results for a four-partite entangled state in the presence of onsite decays as introduced in the main text. The behavior of this measure qualitatively agrees with the results for the concurrence in the two-spin (cf. Fig. 3(b) of the

main text). In particular, we observe the same trade-off between robustness to noise and maximal achievable entanglement.

[1] B. Kraus, H. P. Büchler, S. Diehl, A. Kantian, A. Micheli, and P. Zoller, Phys. Rev. A **78**, 042307 (2008).

[2] M. B. Plenio, S. F. Huelga, A. Beige, and P. L. Knight, Phys. Rev. A **59**, 2468 (1999).

[3] B. Kraus and J. I. Cirac, Phys. Rev. Lett. **92**, 013602 (2004).

[4] S. Clark, A. Peng, M. Gu, and S. Parkins, Phys. Rev. Lett. **91**, 177901 (2003).

[5] A. S. Parkins, E. Solano, and J. I. Cirac, Phys. Rev. Lett. **96**, 053602 (2006).

[6] F. Verstraete, M. Wolf, and J. Cirac, Nature Physics **5**, 633 (2009).

[7] K. G. H. Vollbrecht, C. A. Muschik, and J. I. Cirac, Phys. Rev. Lett. **107**, 120502 (2011).

[8] H. Weimer, M. Müller, I. Lesanovsky, P. Zoller, and H. P. Büchler, Nat. Phys. **6**, 382 (2010).

[9] J. T. Barreiro, M. Müller, P. Schindler, D. Nigg, T. Monz, M. Chwalla, M. Hennrich, C. F. Roos, P. Zoller, and R. Blatt, Nature **470**, 486 (2011).

[10] H. Krauter, C. A. Muschik, K. Jensen, W. Wasilewski, J. M. Petersen, J. I. Cirac, and E. S. Polzik, Phys. Rev. Lett. **107**, 080503 (2011).

[11] H. J. Carmichael, Phys. Rev. Lett. **70**, 2273 (1993); C. W. Gardiner, *ibid.* **70**, 2269 (1993).

[12] C. Kane and M. Fisher, in *Perspectives in the Quantum Hall effect*, edited by S. D. Sarma and A. Pinczuk (Wiley, 1997).

[13] M. Hafezi, E. A. Demler, M. D. Lukin, and J. M. Taylor, Nat. Phys. **7**, 907 (2011).

[14] Z. Wang and S. Fan, Opt. Lett. **30**, 1989 (2005).

[15] L. Feng, M. Ayache, J. Huang, Y.-L. Xu, M.-H. Lu, Y.-F. Chen, Y. Fainman, and A. Scherer, Science **333**, 729 (2011).

[16] J. Koch, A. A. Houck, K. L. Hur, and S. M. Girvin, Phys. Rev. A **82**, 043811 (2010).

[17] A. Kamal, J. Clarke, and M. H. Devoret, Nat Phys **7**, 311 (2011).

[18] For related schemes on two nodes, see [3–5, 10].

[19] K. Stannigel, P. Rabl, A. S. Sørensen, P. Zoller, and M. D. Lukin, Phys. Rev. Lett. **105**, 220501 (2010); G. Anetsberger, O. Arcizet, Q. P. Unterreithmeier, R. Riviere, A. Schliesser, E. M. Weig, J. P. Kotthaus, and T. J. Kippenberg, Nat Phys **5**, 909 (2009).

[20] K. Stannigel, P. Rabl, A. S. Sørensen, M. D. Lukin, and P. Zoller, Phys. Rev. A **84**, 042341 (2011).

[21] This ME stands in contrast to Ref. [10], where the dynamics is purely dissipative with no coherent evolution.

[22] Specifically, $H_A = \sum_i' H_i - i\frac{\gamma}{2}\sum_{j>i}'(c_j^\dagger c_i - c_i^\dagger c_j)$ and $c_A = \sum_i' c_i^-$, where the primed sums run over the nodes of part A, and corresponding expressions hold for B.

[23] O. Astafiev, A. M. Zagoskin, A. A. Abdumalikov, Y. A. Pashkin, T. Yamamoto, K. Inomata, Y. Nakamura, and J. S. Tsai, Science **327**, 840 (2010).

[24] A. V. Akimov, A. Mukherjee, C. L. Yu, D. E. Chang, A. S. Zibrov, P. R. Hemmer, H. Park, and M. D. Lukin, Nature **450**, 402 (2007).

[25] R. J. Schoelkopf and S. M. Girvin, Nature (London) **451**, 664 (2008).

[26] H. J. Kimble, Nature **453**, 1023 (2008).

[27] J. I. Cirac, P. Zoller, H. J. Kimble, and H. Mabuchi, Phys. Rev. Lett. **78**, 3221 (1997).

[28] B. Jungnitsch, T. Moroder, and O. Gühne, Phys. Rev. Lett. **106**, 190502 (2011).