DIMENSIONAL CROSSOVER AND DIMENSIONAL EFFECTS IN QUASI-TWO-DIMENSIONAL BOSE GASES

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This paper gives a systematic review on studies of dimensional effects in pure- and quasi-two-dimensional (2D) Bose gases, focusing on the role of dimensionality in the fundamental relation among the universal behavior of breathing mode, scale invariance and dynamic symmetry. First, we illustrate the emergence of universal breathing mode in the case of pure 2D Bose gases, and elaborate on its connection with the scale invariance of the Hamiltonian and the hidden SO(2, 1) symmetry. Next, we proceed to quasi-2D Bose gases, where excitations are frozen in one direction and the scattering behavior exhibits a 3D to 2D crossover. We show that the original SO(2, 1) symmetry is broken by arbitrarily small 2D effects in scattering, which consequently shifts the breathing mode from the universal frequency. The predicted shift rises significantly from the order of 0.5% to more than 5% in transiting from the 3D-scattering to the 2D-scattering regime. Observing this dimensional effect directly would present an important step in revealing the interplay between dimensionality and quantum fluctuations in quasi-2D.

Keywords: Ultracold gas; dimensional crossover; collective excitation.

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1. Introduction

One of the most fascinating aspects of the many-body system is the role of the system’s dimensionality. Many peculiar quantum phenomena have been known to arise at low dimensions: high-temperature superconductivity, fractional quantum Hall effect, excitations with fractional statistics, topological quantum computation and Berezinskii–Kosterlitz–Thouless (BKT) transition. Compared to the 3D case, dimension reduction alters the density of states and topology of the
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system, leading to dramatically enhanced quantum and thermal fluctuations. These have profound influences on the phase and order of the system. For example, in a 2D homogeneous bosonic system, a true condensate can only exist at zero temperature \((T = 0)\); whereas its formation at finite temperature is strictly prohibited according to the famous Bogoliubov \(k^2\) law or Hohenberg–Mermin–Wagner (HMW) theorem. However, a superfluid phase transition can occur at sufficient low temperature, as was first pointed out by Kane and Kadanoff and then proved by Berezinskii; the 2D XY model undergoes a special type of phase transition to a state which is instead characterized by only algebraic long-range order. This celebrated BKT transition is driven by the unbinding of vortex–antivortex pairs. Such transition has a deep topological origin and has since revolutionized our understanding of the order and phase of matter. Nevertheless, the unambiguous detection of many interesting low-D phenomena remains an outstanding challenge, including observation of BKT transition and the fractional statistics, etc.

Significant advances in cold-atom experiments have opened up a new era of studying many-body quantum systems. Since the realizations of Bose–Einstein condensation (BEC) and of Fermi degeneracy in ultracold dilute gases, the rapid experimental progress with cooling, trapping, and manipulating atoms or molecules have enabled optimal control of the Hamiltonian as well as precise measurement of many-body effects. Recently, the quantum gas microscope for optical lattice adds single site addressing and measurement to the toolbox of possible atomic operations. Building on these experimental advances, quasi-low-D ultracold atomic systems have been created using tight confinement, which sidestep the issues of disorder and decoherence that often plague solid-state physics. Moreover, the experimental possibility to precisely access the system’s energy- and length-scale hierarchy via manipulating the trap frequency and interaction has opened a new area of research on the behavior of quantum gases along the line of dimensional crossovers. For example, BEC and superfluidity generally coexist in 3D; whereas in pure 2D, BKT superfluidity can occur even though BEC is absent. As another example, Efimov state exists in 3D, but not 2D. The interesting question concerning how these distinct behaviors belonging with different dimensions bridge in the crossover regime remains open.

This review specializes to one particular property of a 2D bosonic gas interacting via \(\delta\)-function potential, namely the classical scale invariance. The fundamental consequence of such classical symmetry is the absence of length scale in corresponding dynamics and a universal breathing frequency in a harmonic trap. In the context of quantum field theory, scale invariance and its violation by quantum fluctuations have been well known. On the other hand, the unique behavior of breathing mode has been first noticed by Boltzmann. From the equation which now bears his name, he observed that the breathing mode of a set of particles confined in a 3D isotropic harmonic trap is undamped, irrespective of the ratio between the average collision rate and trap frequency. Similar mode with a universal
frequency has been found in different systems, including a harmonically trapped 2D Bose gas with hard-core interactions,\textsuperscript{35,36} either in its expansion or collective excitations under small perturbations; or a system of trapped particles interacting with a $1/r^2$ potential\textsuperscript{41} where the energy spectrum is divided into a sets of equidistant levels with the separation $2\omega_\perp$. Here, $\omega_\perp$ is the trap frequency in the $x$–$y$ plane. The fundamental origin underlying such universal dynamic property in different systems, as discovered by Pitaevskii and Rosch,\textsuperscript{37} is deeply rooted in the scale invariance of the system’s Hamiltonian and the associated dynamical symmetry described by the 2D Lorentz group $\text{SO}(2,1)$. Ever since, the scale invariance and $\text{SO}(2,1)$-symmetry dictated universality have been widely explored, for example in the unitary fermionic systems.\textsuperscript{42–49} Meantime, violation of the classical symmetry by quantum effects has always been an important subject of study, which provides insight on the quantum many-body effects and the role of dimensionality. Olshanii\textsuperscript{38} has pointed out, for a pure 2D Bose gas, that the classical $\text{SO}(2,1)$ dynamic symmetry does not survive quantization where a regulator scale is introduced and gives rise to dimensional renormalized quantities. From such dimensional transmutation emerges a quantum anomaly which manifests itself as a frequency shift in the breathing oscillation. Hu and Liang\textsuperscript{39} have shown that the confinement-dependent coupling constant of a quasi-2D Bose gas breaks the scale invariance and $\text{SO}(2,1)$ symmetry explicitly along the 3D–2D crossover. In the contexts of Fermi gases, Hofmann\textsuperscript{44} has studied how the hidden $\text{SO}(2,1)$ symmetry of a harmonically trapped 2D Fermi gas was broken by quantum effects. He has shown that such anomalous correction to the symmetry algebra is given by a two-body operator, from which a frequency shift of the breathing mode can be estimated. Meanwhile, Taylor and Randeria\textsuperscript{46} have derived exact results of the monopole breathing mode valid along the entire BCS-BEC crossover and at all temperatures, which is very helpful for understanding why the 2D Fermi gases appears to show scale invariant behavior over a broad range of parameters without the need for fine-tuning.

The plan of current review is to summarize the basic ideas underlying the fundamental relation among the universal breathing mode, the role of dimensionality, the scale invariance, and the $\text{SO}(2,1)$ symmetry in Sec. 2 for pure 2D Bose gases. We describe the hierarchy of dimensional crossover unique of tunable quasi-2D Bose gases created in tight confinement in Sec. 3. In Sec. 4, we discuss the consequence of dimensional effects in the scattering on the scale property and $\text{SO}(2,1)$ symmetry, and derive the induced shift from the universal breathing frequency, all the way from the 3D-scattering to 2D-scattering regime. In the conclusion, we address some of the open problems in the theory of breathing modes in a quasi-2D Bose gas. A brief survey on related work in ultracold fermionic systems is also presented.

2. Pure 2D Bose Gas with $\delta$ Interaction: Classical Scale Invariance, $\text{SO}(2,1)$ Symmetry, and Universal Breathing Mode

In this section, we plan to establish the connections between the classical scale invariance, the dynamical $\text{SO}(2,1)$ symmetry and the universal breathing frequency
for a pure 2D Bose gas interacting with the δ-function potential. We shall first illustrate the emergence of a universal $2\omega_\perp$ breathing frequency by addressing two experimental relevant dynamical problems: (i) the expansion of a 2D Bose gas in Sec. 2.1, (ii) the collective breathing oscillation under external perturbation in Sec. 2.2. The former will be studied based on Gross–Pitaevskii equation and the scaling approach, while the latter will be based on the quantum mechanical equation of motion for the breathing mode operator. Then, we establish the connection of the universal breathing mode with the classical scale invariance of the system in Sec. 2.3, and further show how it is dictated by the dynamic SO(2,1) symmetry in Secs. 2.4 and 2.5. Finally, in Sec. 2.6, we introduce the sum rule approach,\textsuperscript{50–52} which presents a very general method to rigorously find the upper bound to the energy of the breathing oscillation.

### 2.1. Expansion of a pure 2D condensate in time-dependent traps: Exact scaling approach

Consider a pure 2D ultracold Bose gas confined in a symmetric harmonic potential $V_{ho} = \frac{1}{2}m\omega_\perp^2(x^2 + y^2)$. The many-body Hamiltonian\textsuperscript{25,26} describing the model system can be written as, in second quantization,

$$
\hat{H} = \int d^2r \hat{\Psi}^\dagger(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{ho}(r) + \frac{g}{2}\hat{\Psi}^\dagger(r)\hat{\Psi}(r) \right] \hat{\Psi}(r),
$$

where $\hat{\Psi}(r)$ and $\hat{\Psi}^\dagger(r)$ are the boson field operators that annihilate and create a particle at the position $r$, respectively. Here, $g = 4\pi\hbar^2a_s/m$ is a constant coupling constant with $a_s$ being the 3D s-wave scattering length.

We shall formulate our question as follows. (i) Suppose that initially the condensate is in a stationary state described by the stationary Gross–Pitaevskii equation\textsuperscript{25,26} with $\langle \hat{\Psi} \rangle = \Psi_0$,

$$
\left( -\frac{\hbar^2}{2m} \nabla^2 + V_{ho}(r) + g|\Psi_0(r)|^2 - \mu \right) \Psi_0 = 0.
$$

Here, the $\mu$ is the chemical potential of the model system. (ii) Then at $t = 0$, one starts to vary the trap frequency, either adiabatically or suddenly, such that the trap frequency becomes time-dependent as $\omega(t) = \omega_\perp + \delta\omega(t)$. (iii) Finally, the question is how the BEC evolves?

The starting point of solving this dynamic problem is the equation of motion for the field operator $\hat{\Psi}(r, t)$, which can be read off in the Heisenberg picture from its commutation with the many-body Hamiltonian (1), i.e.

$$
i\hbar \frac{\partial}{\partial t} \hat{\Psi} = -\frac{\hbar^2}{2m} \nabla^2 \hat{\Psi} + \frac{m}{2} \omega^2(t)r^2 \hat{\Psi} + g\hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi}^\dagger \hat{\Psi}.$$

Splitting the field operator $\hat{\Psi}$ into a condensate part and a noncondensed one, one writes $\hat{\Psi} = \Psi_0 + \hat{\Psi}'$. Then, Eq. (3) yields the dynamic equations for the $\Psi_0$ and $\Psi'$
in a self-consistent manner,\textsuperscript{53,54}
\begin{equation}
\frac{i\hbar}{\partial t}\psi_0 = -\frac{\hbar^2}{2m} \nabla^2 \psi_0 + \frac{m}{2} \omega^2(t) r^2 \psi_0 + g(|\psi_0|^2 + 2\langle \hat{\psi}^\dagger \hat{\psi} \rangle) \psi_0, \tag{4}
\end{equation}
\begin{equation}
\frac{i\hbar}{\partial t}\hat{\psi}' = -\frac{\hbar^2}{2m} \nabla^2 \hat{\psi}' + \frac{m}{2} \omega^2(t) r^2 \hat{\psi}' + 2g(|\psi_0|^2 + \langle \hat{\psi}'^\dagger \hat{\psi}' \rangle) \hat{\psi}' + g\psi_0^2 \hat{\psi}'^\dagger. \tag{5}
\end{equation}

In 2D, Eqs. (4) and (5) can be exactly solved using the scaling approach.\textsuperscript{36,55}

The basic idea is to transform the variables of \(r\) and \(t\) to the rescaled coordinates \(\rho = r/b(t)\) and time \(\tau(t)\) with \(b(t)\) being the scaling parameter, thereby representing the condensate and noncondensate part \(\Psi_0\) and \(\Psi'\) in a form
\begin{equation}
\Psi_0(r, t) = \frac{\psi_0(\rho, \tau(t))}{b(t)} \exp \left( \frac{i m \rho^2}{2\hbar} \frac{b(t)}{b(t)} \right), \tag{6}
\end{equation}
\begin{equation}
\Psi'(r, t) = \frac{\hat{\psi}'(\rho, \tau(t))}{b(t)} \exp \left( \frac{i m \rho^2}{2\hbar} \frac{b(t)}{b(t)} \right). \tag{7}
\end{equation}

Here, the rescaled time \(\tau\) and scaling parameter \(b\) are determined respectively by
\begin{equation}
\tau(t) = \int_{-\infty}^{t} dt' \frac{1}{b^2(t')}, \tag{8}
\end{equation}
\begin{equation}
\frac{d^2 b(t)}{dt^2} + \omega^2(t) b(t) = \frac{\omega_\perp^2}{b^4(t)}, \tag{9}
\end{equation}
with \(\omega_\perp = \omega(-\infty)\), \(b(0) = 1\) and \(b(0) = 0\). Equations (4) and (5) then transform as
\begin{equation}
\frac{i\hbar}{\partial \tau}\psi_0 = -\frac{\hbar^2}{2m} \nabla^2 \psi_0 + \frac{m}{2} \omega_\perp^2 \rho^2 \psi_0 + g(|\psi_0|^2 + 2\langle \hat{\psi}'^\dagger \hat{\psi}' \rangle) \psi_0, \tag{10}
\end{equation}
\begin{equation}
\frac{i\hbar}{\partial \tau}\hat{\psi}' = -\frac{\hbar^2}{2m} \nabla^2 \hat{\psi}' + \frac{m}{2} \omega_\perp^2 \rho^2 \hat{\psi}' + 2g(|\psi_0|^2 + \langle \hat{\psi}'^\dagger \hat{\psi}' \rangle) \psi_0 + g\psi_0^2 \hat{\psi}'^\dagger. \tag{11}
\end{equation}

Equations (10) and (11) bear a formal resemblance to Eqs. (4) and (5). If one separates the coordinate and time in Eq. (10) as usual, then the same equation as Eq. (2) for the initial stationary condensate is obtained. Hence, Eqs. (10) and (11) are considered as universal in the sense that, in the variables \(\rho\) and \(\tau\), the problem of the nonequilibrium dynamics of quantum many-body systems in time-dependent trap is reduced to an equilibrium one in a fixed harmonic potential. This implies that the equilibrium property of many-body systems can be directly translated to nonequilibrium situations. The reverse conclusion is also true: from a measurement of the system that is out of equilibrium, e.g. a quantum gas after expansion, we can deduce its initial equilibrium properties. In other words, despite the time variation of scaling parameters during pulsation, the physics remains scale independent.

Equation (9) can be used to determine the frequency of breathing mode. Consider a weak modulation of the trap frequency, such that one can write \(b(t) = 1 + f(t) \) with \(f \ll 1\). Then Eq. (9) yields
\begin{equation}
\frac{d^2 f(t)}{dt^2} + 4\omega_\perp^2 f(t) = 2\delta \omega \omega_\perp. \tag{12}
\end{equation}
The solution is
\[ f(t) = \frac{\delta \omega}{2\omega_{\perp}} (1 - \cos 2\omega_{\perp} t), \] (13)
which clearly shows the fundamental frequency of breathing oscillations\(^{35,36}\)
\[ \omega_B = 2\omega_{\perp}. \] (14)

The essential assumption of the scaling approach is that the time-dependence of the model system is entirely contained in the scaling factors, which can be obtained from the evolution of a classical gas. First discussed in the context of a single harmonic oscillator with a time-dependent frequency,\(^{56-61}\) the scaling approach has found wide applications in single-particle problems with potentials of the Coulomb and inverse square type,\(^{60,61}\) as well as many-body problems that include mean-field descriptions of bosonic systems based on the Gross–Pitaevskii equation,\(^{35,36,62-64}\) 1D hard-core bosons\(^{65}\) and in the unitary limit of fermionic gases with infinite scattering length.\(^{42,43}\)

### 2.2. Collective excitation of the breathing mode

As a second example, we consider the excitation of breathing mode in current experiment scenarios. Experimentally, a density perturbation in the BEC can be created by superimposing small AC currents with appropriate phase relationship through magnetic coils. This leads to small modulations of the frequencies of the confining potential. To excite the low-lying collective breathing oscillation, one usually modulates two radial trap frequencies in phase by the same amount \(\delta \omega \ll \omega_{\perp}\). Such action can be described by an excitation operator in a form
\[ F = \frac{m}{2} \sum_i \left\{ [(\omega_{\perp} + \delta \omega)^2 - \omega_{\perp}^2]x_i^2 + [(\omega_{\perp} + \delta \omega)^2 - \omega_{\perp}^2]y_i^2 \right\} \]
\[ \approx m\omega_{\perp} \delta \omega \sum_i (x_i^2 + y_i^2) + o(\delta \omega^2). \] (15)

The excitation (15) creates a density fluctuation \(\delta \rho\) which is proportional to the following quantity
\[ F_B = \sum_i (x_i^2 + y_i^2). \] (16)

The expectation value of \(F_B\) represents the mean-squared radial size of the cloud as a function of time, which is the characteristic quantity describing the pulsation of the condensate. Let us derive the equation of motion for \(F_B\). First, we rewrite the second-quantized Hamiltonian (1) into the form of the first quantization as
\[ H = H_0 + H_{ho} \]
\[ = \sum_i \frac{p_i^2}{2m} + \frac{g}{2} \sum_{i \neq j} \delta(r_i - r_j) + \frac{m}{2} \omega_{\perp}^2 \sum_i (x_i^2 + y_i^2). \] (17)
We then obtain in the Heisenberg picture
\[
\frac{dF_B(t)}{dt} = \frac{i}{\hbar} [H, F_B] = \frac{1}{m} \sum_i (x_i p_{ix} + y_i p_{iy} + \text{h.c.}).
\] (18)

After differentiating Eq. (18) again and evaluating the commutator, we find
\[
\frac{d^2 F_B}{dt^2} + (2\omega_\perp)^2 F_B = \frac{4}{m} H.
\] (19)

It is worth emphasizing that Eq. (19) is an operator identity which is equally valid classically as it is quantum mechanically. Averaging Eq. (19) with respect to some nonequilibrium density matrix, and noting that the energy \(E\) of the system is a constant of the motion, we find
\[
\langle F_B \rangle = A \cos(2\omega_\perp t) + \frac{E}{m\omega_\perp^2}.
\] (20)

In other words, the perturbation (16) generates radial density fluctuations that oscillates at \(2\omega_\perp\).

Compared to the elaborated GP approach in Sec. 2.1, above quantum-mechanical derivation for the breathing oscillation is independent of particular state of the system, and relies only on the Hamiltonian (17) while the statistics does not play a role. This implies specific peculiarities in the behavior of the system, which is connected with certain general feature of the Hamiltonian.

### 2.3. What is scale invariance?

Now let us establish the connection between the breathing oscillation and the scaling property of the system’s Hamiltonian for a 2D Bose gas. Consider a general 2D system without external confinement, which is described by a microscopic Hamiltonian
\[
H_0 = \sum_i -\frac{\hbar^2}{2m} \nabla_i^2 + \sum_{j>i} V(r_i - r_j).
\] (21)

Under a spatial scale transformation
\[
r \to \lambda r,
\] (22)
with \(\lambda\) being the scaling parameter, the Hamiltonian (21) transforms as
\[
H_0 \to \frac{1}{\lambda^2} \sum_i -\frac{\hbar^2}{2m} \nabla_i^2 + \sum_{j>i} V(\lambda r_i - \lambda r_j).
\] (23)

For the \(\delta_2\) potential as in Eq. (1), the scaling property of the interaction\(^{37}\) is
\[
V(\lambda r_i - \lambda r_j) = \frac{1}{\lambda^2} g\delta_2(\lambda r_i - \lambda r_j) = \frac{1}{\lambda^2} V(r_i - r_j).
\] (24)
In other words, under an spatial scaling transformation, the $\delta^2$ interaction potential transforms in exactly the same way as the kinetic energy does. In such case, the scaling for the Hamiltonian follows the $\lambda^{-2}$ law, i.e.

$$H_0 \rightarrow \frac{1}{\lambda^2} H_0.$$  \hfill (25)

A fundamental consequence of such scale invariance is the absence of any length scale in corresponding dynamics. More precisely, if $\psi$ is an eigenstate of energy $E$ in free space, then $\psi_\lambda(r) = \psi(\lambda r)$ is an eigenstate of energy $E/\lambda^2$ for any $\lambda$.

In order to see the implication of scaling invariance on the breathing dynamics in a homogeneous harmonic trap, let us return to the quantity (15). At the classical level, $F_B$ is connected to the moment of inertia of the system, while

$$Q = \sum_i r_i \cdot p_i / m$$

is known as the scaler virial. For a general interaction potential $V(\lambda r_i) = \lambda^n V(r_i)$ that gives rise to a power-law type of force, it follows from the virial theorem in classical mechanics that

$$\partial_t Q = 2T - nV - 2V_{ho}.$$  \hfill (27)

For a potential with scaling index $n = -2$, Eq. (27) for $V_{ho} = (1/2)\omega_\perp^2 r_i^2$ immediately yields the equation $\partial^2 F_B / \partial t^2 + 4\omega_\perp^2 F_B = 2E$ with the obvious solution $F_B = A \cos(2\omega_\perp t + \theta) + E/(m\omega_\perp^2)$. As is manifestly shown, the existence of $2\omega_\perp$ is therefore connected with the $n = -2$ scaling of the interacting potential.

2.4. Dynamic SO(2, 1) symmetry and the universality

As is discovered by Pitaevskii and Rosch,\textsuperscript{37} the universal $2\omega_\perp$ breathing frequency is dictated by a specific symmetry property connected with the scaling invariance, namely the dynamic SO(2, 1) symmetry. The unique role of a harmonic trap consists in its commutation with the Hamiltonian which provides the generator of scale transformation, namely

$$[H_{ho}, H] = i\omega_\perp^2 Q,$$ \hfill (28)

with $Q$ being defined as

$$Q = \sum_i (p_i \cdot r_i + r_i \cdot p_i) / 2.$$ \hfill (29)

Note that the quantity $Q$ at the classical level is just the scaler virial in Eq. (27).

Consequently, following algebra can be established

$$[H_{ho}, H] = i\omega_\perp^2 Q, \quad [Q, H_0] = 2iH_0, \quad [Q, H_{ho}] = -2iH_{ho}.$$ \hfill (30)

Equivalently, one defines

$$L_1 = \frac{1}{2\omega_\perp}(H_0 - H_{ho}), \quad L_2 = \frac{Q}{2}, \quad L_3 = \frac{1}{\omega_0} H.$$ \hfill (31)
which leads to the well-known SU(1, 1) or SO(2, 1) algebra

\[ [L_1, L_2] = -iL_3, \quad [L_2, L_3] = iL_1, \quad [L_3, L_1] = iL_2. \]  

(32)

Equipped with above algebra of SO(2, 1), the existence of universal breathing frequency \( 2\omega_\perp \) becomes obvious if one defines \( L^\pm = (1/\sqrt{2})(L_1 \pm iL_2) \) which immediately yields

\[ [H, L^\pm] = \pm 2\omega_\perp L^\pm. \]  

(33)

Thus, starting from the lowest eigenstate of Hamiltonian \( H \) with the ground state energy of \( E_0 \), i.e. \( H |\Psi_0\rangle = E_0 |\Psi_0\rangle \), one can construct higher states with energies \( E_0 + n(2\omega_\perp) \), with \( n = 1, 2, \ldots \) by applying \( HL^\dagger |\Psi_0\rangle = (E_0 + 2\omega_\perp)L^\dagger |\Psi_0\rangle \). Apparently, an infinite number of excitations with energies \( n(2\omega_\perp) \) exists, which we identify with the breathing modes of the model system.

2.5. Symmetry and beyond

The infinite tower of breathing modes studied in Sec. (2.4) is in fact a general consequence of nonrelativistic conformal invariance.\(^{45,47}\) We briefly review the useful Schrödinger algebra as follows:

\[ C = \int dr \frac{\Psi^\dagger \Psi}{\omega_\perp}, \quad K = \int dr \Psi^\dagger \Psi, \quad P = \int dr J(r) \]  
\[ D = \int dr \cdot J(r) \]

with \( J = -im/2\hbar \partial \). Then we can construct the operators\(^{47}\)

\[ M^\dagger = \frac{1}{2} \left( \frac{P}{\omega_\perp} + i\sqrt{\omega_\perp}K \right), \]  

(34)

\[ L^\dagger = \frac{1}{2} \left( \frac{H_0}{\omega_\perp} - \omega_\perp C + iD \right). \]  

(35)

With the help of \( [H, M^\dagger] = \omega_\perp M^\dagger \) and \( [H, L^\dagger] = 2\omega_\perp L^\dagger \), one can easily check that the operators of \( M^\dagger \) and \( L^\dagger \) excite the center-of-mass energy eigenstates and breathing eigenstates, respectively, by acting repeatedly on the \( N \)-body primary state. While for \( N = 1 \) the unique primary state is the ground state of the total Hamiltonian, for \( N \geq 2 \) there are an infinite number of primary states. The energy spectrum is thus organized in infinite ladders with a ladder built on top of every primary state. While the individual breathing states do not actually deform in time because they are the eigenstates of the total Hamiltonian, the time evolution of a linear combination of the states from a given ladder produces breathing density oscillation\(^{47}\) decomposable into modes with frequencies of \( n(2\omega_\perp) \).

Note that \( L^\dagger \) excites both internal and center-of-mass degrees of freedom because of \( [L^\dagger, M^\dagger] = -M^\dagger \neq 0 \). Due to separability of the center-of-mass and internal motion in a harmonic trap, one can construct the operator \( B^\dagger = L^\dagger - M^\dagger \cdot M^\dagger /2mN \), which excites internal breathing eigenstates.\(^{47}\) We can calculate \( [H, B^\dagger] = 2\omega B^\dagger \) and \( [B^\dagger, M^\dagger] = [B^\dagger, M] = 0 \), which means that the operator \( B \) is the proper generator of internal breathing modes in a harmonic trap.
2.6. Sum rule approach

The sum rule approach\textsuperscript{50–52} has provided a powerful and rigorous approach for analysis of collective excitations of many-body systems, capitalizing on such general features of the system as energy conservation and virial theorem, etc. Experimentally, the breathing mode can be excited by modulating the trapping frequency $\omega_{\perp}$. Therefore, in the excitation spectrum, the breathing mode corresponds to a sharp peak, which is fully characterized by the dynamic structure factor

$$S(\omega) = \sum_f |\langle f | F_B | g \rangle| \delta(\hbar \omega - E_f + E_g).$$

The key quantity in sum rule technique is the energy-weighted moments\textsuperscript{66}

$$m_p = \int d\omega S(\omega) \omega^p. \quad (36)$$

In particular, ratios between different sum rules provide rigorous upper bounds of the breathing mode $\omega_B$ as follows:

$$\hbar \omega_{p,p-2} = \sqrt{\frac{m_p}{m_{p-2}}}. \quad (37)$$

Usually, one adopts $m_3/m_1$ or $m_1/m_{-1}$. Let us illustrate with

$$m_3 = \frac{1}{2} \langle [[F_B^\dagger, H], [H, [H, F_B]]] \rangle = \frac{2}{m^2} \langle [[Q, H], Q] \rangle, \quad (38)$$

$$m_1 = \frac{1}{2} \langle [[F_B, H], F_B] \rangle. \quad (39)$$

Consider a system of $N$ particles that has a total energy $E = T + V + V_{ho}$, where $T$, $V$ and $V_{ho}$ refers to the kinetic energy, interaction potential and harmonic trap energy, respectively. For $V_{ho} = \frac{1}{2} m \omega_{\perp}^2 (x^2 + y^2)$, we obtain

$$\frac{m_3}{m_1} = \frac{8 \hbar^4}{m^2} N (T + V + V_{ho}) \frac{2 \hbar^2 \omega_{\perp}^2}{4 \hbar^2 N m^2 \omega_{\perp}^2 V_{ho}} = 2 \hbar^2 \omega_{\perp}^2 \left(1 + \frac{T + V}{V_{ho}} \right). \quad (40)$$

If we invoke the 2D varial theorem, i.e.

$$T + V - V_{ho} = 0, \quad (41)$$

one ultimately obtains

$$\omega = \frac{1}{\hbar} \sqrt{\frac{m_3}{m_1}} = 2 \omega_{\perp}. \quad (42)$$

Above derivation does not rely on the the microscopic details of the system, again indicating the connection between the universal $2 \omega_{\perp}$ frequency with certain general features of the system.
3. Quasi-2D Bose Gases: Dimensional Crossover in Scattering, Broken Scale Invariance, and Breathing Mode

3.1. Basic picture of the hierarchical dimensional crossover

In experimental scenarios, creation of 2D Bose gases is achieved via hierarchical access to new energy and length scales by carefully manipulating the axial trap and the scattering length. The basic picture of such dimensional crossover can be understood as follows.

- First, Bose gases in tight axial confinement become quasi-2D when energetic restriction to freeze axial excitations is reached. At this stage, particle motion in the trap direction is confined to zero point oscillations, and kinematically the gas becomes 2D. Experimentally, quasi-2D quantum degenerate Bose gases have been produced in single pancake traps or at the nodes of one-dimensional optical lattice potential.

- Next, into the quasi-2D regime, the restricted kinematics due to tight confinement gives rise to a new length scale \( \sigma_z \), which characterizes the extension of the wavefunction along the frozen direction.

- The new length scale \( a_{2D} = \sigma_z \) competes with the 3D scattering length \( a_{3D} \) in the two-particle scattering process. As a consequence, two distinctive scattering regimes can be identified: (i) in the regime \( a_{3D}/a_{2D} \gg 1 \), the wavefunctions of the two colliding particles retain the 3D geometry in spite of slight shape modulations by the trap, therefore the scattering is effectively 3D; (ii) in the opposite \( a_{3D}/a_{2D} \ll 1 \), the single-particle wavefunction has an axial extension so small compared to the scattering length that the geometry of the wavefunction asymptotically approaches 2D, hence the collisions can be effectively viewed as occurring between two 2D particles; in this 2D-scattering regime, the corresponding coupling constant shows a characteristic strong density dependence. As a result, when accessing from one extreme of \( a_{3D}/a_{2D} \) to the other in a quasi-2D gas, a further dimensional crossover emerges in the behavior of bi-atomic collisions or binary scattering interactions.

3.2. Confinement-dependent coupling constant in quasi-2D gas

Let us elaborate on the microscopic description of a quasi-2D Bose gas by considering a typical experiment scheme where a 1D optical lattice along the \( z \)-direction is applied

\[
V_{\text{opt}} = sE_R \sin^2(q_B z).
\]  

(43)

Here \( s \) is a dimensionless factor labeled by the intensity of a laser beam and \( E_R = \hbar^2 q_B^2 / 2m \) is the recoil energy with \( \hbar q_B \) being the Bragg momentum. The lattice period is fixed by \( q_B = \pi / d \) with \( d \) being the lattice spacing. Atoms are unconfined in the \( x-y \) plane.
The quasi-2D regime is achieved when the energetic restriction $\mu \gg 4t$ is fulfilled, with the $\mu$ and $t$ being the chemical potential and the tunneling rate between two wells respectively. In this limit, the tunneling is negligibly small compared to the tight confinement, and the system consists of an array of pancakes which can be effectively modeled by the Hamiltonian

$$H_{Q2D} = \sum_j -\frac{1}{2m} (\nabla_j^2 + \nabla_j^2) + g_{Q2D} \sum_{j<k} \delta^2(r_{jk}).$$

(44)

The key assumption underlying Eq. (44) is that the confined axial degrees of freedom along the lattice direction are taken into account via a kinematically renormalized coupling constant $g_{Q2D}$ that is expressed as

$$g_{Q2D} = \frac{2\sqrt{2\pi} \hbar}{m} \frac{1}{a_{2D}/a_{3D} + 1/\sqrt{2\pi} \log[B/\pi n_{2D}^2]},$$

(45)

with $B = 0.915$ and $n_{2D} = nd$ being the surface density. Here, the identification of $a_{2D} = \sigma_z$ and $a_{3D} = a$ has been made. The existence of two length scales in the coupling constant is immediately evident from Eq. (45). The consequences are: (a) the appearance of a log density-dependent correction as a manifestation of 2D effects in scattering as shown in Eq. (46), its strength being measured by the ratio $a_{3D}/a_{2D}$, and (b) the identification of two distinctive scattering regimes: the 3D-scattering regime $a_{3D}/a_{2D} \ll 1$ with weak 2D effects and the 2D-scattering one $a_{3D}/a_{2D} \geq 1$ where the kinematic modification of the coupling constant is important. In the extreme $a_{2D}/a_{3D} \rightarrow 0$, Eq. (45) becomes independent of the value of $a_{3D}$ and a regime of purely 2D scattering is achieved. In this limit, Eq. (45) asymptotically approaches the coupling constant of a purely 2D Bose gas, reading

$$g_{2D} = \lim_{a_{2D}/a_{3D} \rightarrow 0} g_{Q2D} = \frac{4\pi \hbar^2}{m} \frac{1}{\log[1/n_{2D}^2 a_{2D}^2]}.$$

(46)

Here the logarithmic dependence on the gas parameter $n_{2D} a_{2D}^2$ is unique of the 2D geometry.

### 3.3. Ground state energy of quasi-2D Bose gas along the dimensional crossover

The ground state energy of an optically quasi-2D Bose gas can be obtained starting from the grand partition function$^{83,84}$ $Z = \int D[\psi^*, \psi] e^{-S[\psi^*, \psi]}$, where

$$S[\psi^*, \psi] = \int d\tau \int d^3r \psi^*(r, \tau) \left[ \hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2 \nabla^2}{2m} + V_{opt}(r) + \frac{g_e}{2} |\psi(r, \tau)|^2 \right] \psi(r, \tau)$$

(47)

is the action functional of $[\psi^*(r, \tau), \psi(r, \tau)]$ which collectively denote the complex functions of space and imaginary time $\tau$. Here, $g_e$ abstractly stands for the two-body coupling constant in an axial optical confinement. Using the path-integral
approach,\textsuperscript{83,84} one finds within the tight-binding approximation and Bogoliubov theoretical framework\textsuperscript{85,86} the ground state energy $E_g$,

$$
\frac{E_g}{V} = \frac{1}{2} g_c n^2 \left[ 1 + \frac{m \tilde{g}_c}{2 \pi^2 \hbar^2 d} F \left( \frac{2t}{\tilde{g}_c n} \right) \right],
$$

where $\tilde{g}_c = g_{Q2D} [d \int_{-d/2}^{d/2} w^4(z) dz] = g_{Q2D} \sqrt{2\pi \sigma}$, with $w(u) = \exp[-u^2/2\sigma^2]/\pi^{1/2}\sigma^{1/2}$ being a variational Gaussian ansatz and the ratio $d/\sigma \simeq \pi^{1/4} \sigma^{1/2}$ obtained from minimizing the free energy functional with respect to $\sigma$.\textsuperscript{87} The function in Eq. (48) reads

$$
F(x) = \frac{(x+1)}{2} \left[ (3x+1) \arctan \left( \frac{1}{\sqrt{x}} \right) - 3 \sqrt{x} \right] - \frac{\pi}{2} \ln \left( \frac{x}{2x+1+2 \sqrt{x(x+1)}} \right)

- \pi \text{arcsinh}(\sqrt{x}) + 2 \int_{0}^{\sqrt{x}} \frac{\tan^{-1}(z)}{z} dz.
$$

(49)

Here, $t$ denotes the tunneling rate, $n$ refers to the condensate density.

The basic physical picture captured by the expression of ground state energy in Eq. (48) concerns the 3D–2D crossover in the kinematics of the system which is governed by the parameter $2t/\tilde{g}_c n$. In the limit $2t/\tilde{g}_c n \gg 1$, one finds the system exhibiting anisotropic 3D behavior and $F(x) \simeq 32/15 \sqrt{x}$. Whereas, the extreme $2t/\tilde{g}_c n \ll 1$ corresponds to the quasi-2D regime where axial atomic motion is frozen to zero-point oscillations and $F(x) = -\frac{\pi}{4} - \frac{\pi}{2} \log x$ is approached exactly. Substituting $F(x) = -\frac{\pi}{4} - \frac{\pi}{2} \log x$ into Eq. (48), we arrive at the ground state energy expression for a quasi-2D Bose gas,

$$
\frac{E_g}{V} = \frac{1}{2} g_c n^2 \left[ 1 + A \tilde{g}_c \left( \frac{\pi}{4} - \frac{\pi}{2} \log \left[ 2t/\tilde{g}_c n \right] \right) \right].
$$

(50)

We stress that the above analysis is justified by the weak but nonnegligible tunneling effect guaranteed by $1/N_t \leq t/\tilde{g}_c n$ with $N_t$ being the number of atoms per optical well.

### 3.4. Broken scale invariance in quasi-2D Bose gases

Let us go back to the microscopic Hamiltonian of a quasi-2D Bose gas in Eq. (44) and analyze its scale property. Under the transformation $r \rightarrow \lambda r$,

$$
H_{Q2D} \rightarrow \frac{1}{\lambda^2} \sum_j \frac{1}{2m} \left( \nabla^2_{jx} + \nabla^2_{jy} \right) + g_{Q2D}(n_{Q2D}(\lambda r)) \frac{1}{\lambda^2} \sum_{j<k} \delta^2(r_{jk}).
$$

(51)

As is manifestly shown, because of the explicit density dependence of $g_{Q2D}$ due to dimensional effects in scattering, a quasi-2D Bose gas does not possess exact scale invariant symmetry.

In most experiments on 2D quantum gases to date,\textsuperscript{67,88–90} a small ratio $a_{3D}/a_{2D} \leq 0.05$ is usually pursued. In this case, according to Eq. (45), the density-dependent part only consists of a 2% contribution to $g_{Q2D}$ and is typically ignored.
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Accordingly, the 2D coupling was frequently evaluated as $g_{Q2D} = \sqrt{8\pi\hbar^2a_{3D}/ma_{2D}}$. One may thereby tend to conclude that a quasi-2D Bose gas is scale invariant. Nevertheless, we stress that (i) the 2D effects in $g_{Q2D}$, however small it is, will fundamentally break the scale invariance of $H_{Q2D}$ in Eq. (51). (ii) the 2D effects are contained in higher order of $a_{3D}/a_{2D}$, which become important in the regime $a_{3D}/a_{2D} > 1$ and can not be ignored.

It is also interesting to compare the breaking of scale-invariance in quasi-2D Bose gases by dimensional effects to that of quantum anomaly. In pure 2D, the $\delta$-function interaction is not well defined due to logarithmic ultraviolet divergences. The subsequent regularization forces a cut off by the finite range $r_e$ of the interaction,\textsuperscript{80} which introduces a regulator scale into the original system. The renormalized coupling constant thereby becomes dimensionful. As a result, the original classical scale invariance of a pure 2D Bose gas is broken even for constant density, and a quantum anomaly emerges.

4. Visualizing Dimensional Effects in Collective Breathing Mode

As a direct consequence of broken scale invariance due to dimensional effects in scattering, the quasi-2D Bose gas does not strictly possess the SO(2,1) symmetry. Alternatively, the shift of breathing frequency from the universal frequency $2\omega_\perp$ provides crucial information on the effects of dimensionality.

In what follows, we present a detailed derivation of the collective excitations of a quasi-2D Bose gas tightly confined by an optical lattice $V_{\text{opt}} = sE_R\sin^2(q_Bz)$, with a superimposed cylindrically symmetric harmonic trap $V_{ho}(r) = \frac{m}{2}(\omega_\perp^2x^2 + \omega_z^2y^2 + \omega_2^2z^2)$, as shown in Fig. 1.

![Fig. 1. Schematic picture of an optically trapped quasi-2D Bose gas. A 1D optical lattice is along the horizontal axis (z-axis) with a BEC in an elongated harmonic trap, with axial (radial) frequency $\omega_z$ ($\omega_\perp$). The BEC is thus confined to an array of narrow potential pancakes.](1330010-14)
4.1. Hydrodynamics of a quasi-2D Bose gas

Our starting point is the linearized hydrodynamic equation for density fluctuations \( \delta n (r, t) \) for quasi-2D Bose gas created in a 1D optical confinement,91,92 in the presence of an additional harmonic trap

\[
m \frac{\partial^2 \delta n}{\partial t^2} - \nabla \cdot \left[ n \nabla \left( \frac{\partial \mu_{Q2D}}{\partial n} \delta n \right) \right] = 0 ,
\]

(52)

with \( \nabla \equiv (\nabla_\perp, \nabla_z \sqrt{m/m^*}) \). Note that the effect of optical lattice is further reflected in the renormalized mass \( m^* \) along the lattice direction. Equation (52) is justified by sufficiently weak tunneling which is nevertheless nonnegligible to ensure full coherence of the order parameter between different wells,91 and by assuming the Thomas–Fermi (TF) limit and local density approximation.25,26

At the core of the hydrodynamic Eq. (52) is the zero-temperature local equation of state \( \mu_{Q2D} \) of the quasi-2D Bose gas under consideration. The 3D density \( n(r) \) is then determined from \( \mu_0 = \mu_{Q2D}[n(r)] + V_{ho}(r) \), where \( \mu_0 \) is the ground state value of the chemical potential, fixed by the proper normalization of \( n(r) \). In what follows, we analyze the equation of state \( \mu_{Q2D} \) in the 3D scattering regime \( a_{3D}/a_{2D} \ll 1 \) and 2D-scattering regime \( a_{3D}/a_{2D} \geq 1 \), respectively. In the end, we would like to understand how the dimensional effects in scattering can shift the breathing mode away from its universal frequency.

4.2. Derivation of breathing frequency in the 3D-scattering regime

The \( \mu_{Q2D} \) can be readily determined via \( \mu = \partial E_g/\partial N \) from Eq. (50), together with proper asymptotic analysis. To begin with, we first note that in the asymptotic 3D regime where \( 2t/\tilde{g}n \gg 1 \) and \( a_{3D}/a_{2D} \ll 1 \), our analytical solution \( \mu = \tilde{g}n [1 + (32m^*/3\sqrt{\pi m}) \sqrt{a_{3D}^2 n}] \) is consistent with the 3D Lee–Huang–Yang (LHY) result;93–95 whereas, in the opposite pure 2D limit where \( 2t/\tilde{g}n \ll 1 \) and \( a_{3D}/a_{2D} \gg 1 \), the asymptotical value for the chemical potential of a 2D Bose gas

\[
\mu = \frac{4\pi\hbar^2 n_{2D}/m}{\ln (1/n_{2D}a_{2D}^2)} \left[ 1 - \frac{\ln (1/n_{2D}a_{2D}^2)}{\ln (1/n_{2D}a_{2D}^2)} \right] \left[ 1 - \frac{\ln (1/n_{2D}a_{2D}^2)}{\ln (1/n_{2D}a_{2D}^2)} \right] - B \ln (1/n_{2D}a_{2D}^2) - B \ln (1/n_{2D}a_{2D}^2)
\]

with \( B = 1 - \ln (m_t/n_{2D}2\pi\hbar^2) \) stands in good agreement with previous studies.72

We are now ready to generalize above analysis to the quasi-2D regime \( 2t/\tilde{g}n \ll 1 \) with 3D scattering \( a_{3D}/a_{2D} \ll 1 \). Since \( a_{3D}/a_{2D} \) is a small parameter, we adopt the perturbation technique to analyze the 2D effects on the breathing mode. Expanding Eq. (45) to linear order of \( a_{3D}/a_{2D} \), we write

\[
\tilde{g}_e = \tilde{g} \left[ 1 - \frac{1}{\sqrt{2\pi}} \frac{a_{3D}}{a_{2D}} \ln \left( \frac{1}{n_{2D}a_{2D}^2} \right) \right] .
\]

(53)
Substituting Eq. (53) into Eq. (50) and differentiate with $N$, we obtain
\[ \mu_{Q2D} = \tilde{g}n \left[ 1 + \frac{1}{\sqrt{2\pi} a_{2D}} \left( \frac{1}{2} - \ln \left( \frac{1}{n_{2D}a_{2D}^2} \right) \right) \right]. \] (54)

Rewriting $\mu_{Q2D} = \tilde{g}n[1 + k_{2D}(n)],$ we identify $k_{2D}(n) = a_{3D}/(\sqrt{2\pi}a_{2D})[1/2 - \ln(1/n_{2D}a_{2D}^2)]$ as the first correction to the 3D mean-field (MF) equation of state arising from the 2D effect.

From Eq. (54), the 3D ground state density can be solved by iteration
\[ n(r) = n_{TF} - \frac{1}{\sqrt{2\pi} a_{2D}} \left[ \frac{1}{2} + \ln \left( \frac{1}{dn_{TF}a_{2D}^2} \right) \right] n_{TF}. \] (55)

with $n_{TF}(r) = (\mu_0 - V_{ext}(r))/\tilde{g}$ being the 3D TF density. Equation (55) clearly shows that the 2D corrections in the local chemical potential in Eq. (54) are transferred to the 3D stationary shape of cloud.

Substituting Eqs. (54) and (55) into Eq. (52) and only retaining terms linear in $k_{2D}(n)$, we obtain
\[ m\omega^2\delta n + \nabla \cdot (\tilde{g}n_{TF} \nabla \delta n) = -\nabla^2 \left( \tilde{g}n_{TF}^2 \frac{\partial k_{2D}}{\partial n_{TF}} \delta n \right). \] (56)

Equation (56) in the absence of $k_{2D}$ is just the familiar 3D hydrodynamic equation in the presence of 1D optical lattice.\(^{25,26}\) Against this background, the addition of terms on the right side of Eq. (56) presents a perturbation. The resulting fractional frequency shift, to the leading order, is given by
\[ \frac{\delta \omega}{\omega} = -\frac{\tilde{g}}{2m\omega^2} \int d^3r \nabla^2 \delta n^* \left( \frac{n_{TF}^2}{dn_{TF}} \frac{\partial k_{2D}}{\partial n_{TF}} \delta n \right), \] (57)

where integrals extend to the region where $n_{TF}$ is positive.\(^{25,26}\) An important feature in Eq. (57) is the dependence of $\delta \omega/\omega$ on the derivative $\partial k_{2D}/\partial n$ rather than $k_{2D}(n)$. The consequence is that the leading order correction arising from 2D effect to 3D MF collective frequency shows no logarithmic density dependence.

According to Eq. (57), the surface modes that satisfy $\nabla^2 \delta n = 0$ are not perturbed by the 2D effect in the vicinity of 3D regime. Hence, in order to observe dimensional effects, one has to focus on small compressional oscillations. Our primary mode of interest is the transverse breathing mode in a very elongated trap ($\sqrt{m/m^\perp \omega_z/\omega_\perp} \ll 1$). Substitutions of $\delta n(r) \sim r_\perp^2 - R_{TF}^2/2$ with $R_{TF} = \sqrt{2\mu_0/m\omega_z^2}$ being the transverse TF radius and $\omega = 2\omega_\perp$ into Eq. (57) yield the fractional shift
\[ \frac{\delta \omega}{\omega} = \frac{1}{4\sqrt{2\pi} a_{2D}}. \] (58)

In typical experiments to date,\(^{73}\) the relevant parameters are given by $a_{3D} = 5.31$ nm and the lattice period $d = 297.3$ nm. The frequency shift in Eq. (58) can be reached $\sim 0.48\%$ for $s = 4$. Given an accuracy of $\sim 0.3–0.4\%$ in measuring collective frequencies within current facilities,\(^{22}\) the 2D correction to the transverse breathing
mode is well in reach in relevant experiment conditions. Moreover, this effect can be enhanced via adjusting lattice parameter and using Feshbach resonance.\(^{24}\) We have also taken a look at the lowest compression mode in a disk-like geometry \((\sqrt{m/m^*}\omega_z/\omega_\perp \gg 1)\), which is along the axial direction with the zeroth order dispersion given by \(\omega = (\sqrt{m/m^*})\sqrt{3}\omega_z\) and density oscillations of the form \(\delta n(r) \sim z^2 - Z_{TF}^2/3\), where \(Z_{TF} = \sqrt{2m^*\mu_0/m^*_z}\) is the TF radius along the axial direction.

Straightforward calculations yield \(\delta \omega/\omega = 1/6\sqrt{2\pi m^*/m^*_a 3D/m^*_a 2D}\), showing an amplified 2D effect due to the increased inertia along the direction of the laser.

### 4.3. Derivation of breathing mode in 2D-scattering regime

Now, let us turn to the 2D-scattering regime of a quasi-2D Bose gas \(a_{3D}/a_{2D} \geq 1\) and \(2t/\tilde{g}n \gg 1\). In this regime, the role of 2D kinematics becomes pronounced in scattering process, resulting in strong log density dependence of \(g_{Q2D}\). Hence, previous perturbation expansion with respect to \(a_{3D}/a_{2D}\) no longer applies. Moreover, the effect of quantum fluctuations becomes significant compared to the 3D-scattering-dominated regime.\(^{70}\) We thereby retain the concrete form of \(g_e\), as well as fully account for the effect of quantum fluctuations in the ground state energy (50) when deriving the equation of state. The result is, to the second order in \(\tilde{g}_e\),

\[
\mu_{Q2D} = \tilde{g}_e n + \frac{A\pi}{2} n \tilde{g}_e^2 - \frac{A\pi}{2} n \tilde{g}_e^2 \log \left( \frac{2t}{\tilde{g}_e n} \right),
\]

(59)

where \(A = m/2\pi^2\hbar^2d\).

We adopt the scaling approach introduced in Sec. 2.1, and solve Eq. (52) with the rescaled density and velocity:\(^{35,36}\)

\[
n = n_0(r_i/\lambda_i)/\Pi_j \lambda_j, \\
v_i = (\lambda_i/\lambda_0)r_i \quad (i = x, y, z),
\]

(60)

The equation for the scaling parameter \(\lambda_i\) in the transverse and axial direction read, respectively,

\[
d^2 \lambda_\perp/dt^2 + \omega_\perp^2 \lambda_\perp + (\omega_\perp^2/\lambda_\perp)F(\lambda_\perp, \lambda_z) = 0, \\
d^2 \lambda_z/dt^2 + \omega_z^2 \lambda_z + (\omega_z^2/\lambda_z)F(\lambda_\perp, \lambda_z) = 0,
\]

(61)

(62)

with initial condition \(\lambda_i(0) = 1\) and \(\dot{\lambda}_i(0) = 0\). Here,

\[
F = \frac{1}{m(\omega_i)^2(r_i/\lambda_i)} \frac{1}{\lambda_i} \int n(r_0, 0)r_i \frac{\partial \mu_{Q2D}}{\partial r_i} dr_i.
\]

(63)

From the initially equilibrium condition, one finds \(F(1) = -1\) and \(F(\lambda_\perp, \lambda_z)\) independent of the particular coordinate \(i\). Linearizing Eqs. (61) and (62), one finds that the breathing mode for a quasi-2D boson gas in a very elongated trap...
\[\sqrt{m/m^*} \omega_z/\omega_\perp \ll 1,\]

\[
\frac{\delta \omega}{\omega_B} = \left[ 1 + \frac{1}{\sqrt{2\pi^3}} \frac{a_{2D}}{a_{3D}} + \frac{1}{\sqrt{2\pi}} \log \left[ B\hbar\omega_0/\pi \mu \right] \right]^{1/2} - 1, \tag{64}
\]

with \( B = 0.915 \), and \( \hbar\omega_0/\mu = 1/\pi n(0)a_{2D}^2 \). Here, \( n(0) \) being the density at the center of harmonic trap.

In typical experiments,\(^67\) the gas is deep in the quasi-2D regime for \( \hbar\omega_0/\mu = 6 \). Hence by setting \( \hbar\omega_0/\mu = 6 \) in Eq. (64), we estimate the fractional shift as \( \delta \omega/\omega_B \sim 5\% \) for \( a_{3D}/a_{2D} = 1 \). Compared to the 3D-scattering regime where \( \delta \omega/\omega_B \sim 0.5\% \) from Eq. (58), the frequency shift raises significantly in the 2D scattering regime, indicating a marked deviation from the 3D behavior in scattering process.

To conclude this section, we mention that in most experiments to date,\(^67,88\) \( a_{3D}/a_{2D} \ll 1 \) is typically realized, and the created quasi-2D Bose gases are in the 3D-scattering regime. In this regime, the frequency shift in breathing mode is given by Eq. (58), which is typically \( \delta \omega/\omega_B \sim 0.5\% \). This imposes a high accuracy of \( \leq 0.3\% \) in measuring collective frequencies, in order to observe the frequency shift induced by dimensional effects. In contrast, the frequency shift in the 2D-scattering regime is given by Eq. (64) and is of the order 5\%, which is well in reach in current facilities. However, an experimental ability to produce larger ratio of \( a_{3D}/a_{2D} \) is required. Such large ratio can be achieved for Cs atoms and some experiments are under way according to our knowledge.

### 4.4. Experimental observation of breathing mode

In this section, we briefly summarize the experimental progress in measuring the frequency of breathing oscillation in various ultra-cold atomic systems.

Creation of 2D atomic gases requires freezing the motion along the \( z \)-direction using either light-induced forces or magnetic forces.\(^28\) Such a confining potential has to be strong enough, so that all relevant energies are well below the excitation energy from the ground state to the first excited state. The two other directions \( x \) and \( y \) are much more weakly confined.

The long-lived breathing mode was first observed in a 3D quasi-cylindrical geometry by Chevy et al.,\(^88\) rather than in a pure 2D geometry. In such experiment, the transverse breathing mode was found to oscillate at a frequency very close to \( 2\omega_\perp \); an extremely small damping is found with quality factor of the mode \( > 2000 \). Then Stock et al.\(^89\) has observed the breathing mode in a fast rotating gas. Its frequency was also around \( 2\omega_\perp \), with a small correction due to the nonharmonicity of the trapping potential that was necessary to stabilize the center-of-mass motion of the atom cloud in the fast rotating regime. The first observation of the breathing mode with \( 2\omega_\perp \) in a Bose gas confined in the 2D geometry with optical lattice was achieved by Fort et al.\(^90\) They have concluded that the breathing
mode along the radial direction was not affected by the presence of the optical lattice. Furthermore, Hung et al.,\textsuperscript{67} experimentally verified the scale invariance and university of an interacting 2D Bose gas by observing in situ density and density fluctuations of 2D \(^{133}\)Cs gases at various temperatures. In such experiment, the ratio \(\hbar \omega_z/\mu_0 > \hbar \omega_z/k_B T > 6\) indicates that the BEC sample is deep in the 2D regime with only < 1% population at the vertical excited states. In a recent experiment of 2D Fermi gases by Vogt et al.,\textsuperscript{96} the breathing mode frequency has been measured to be around \(2 \omega_\perp\) at temperature \(T \approx 0.4 T_F\), with \(T_F\) being the Fermi temperature.

To our best knowledge, in all the experiments so far, no substantial deviation from the universal frequency \(2 \omega_\perp\) of breathing mode have been observed, both in the contexts of Bose and Fermi gases. The broken scale invariance in a quasi-2D gas due to dimensional effects in scattering, as well as the existence of quantum anomaly, remain to be testified.

5. Conclusion

In this brief review, we intended to give a self-contained review regarding the fundamental relation among the role of dimensionality, the peculiar behavior of the breathing mode, the scale properties of the system and the SO(2, 1) symmetry. In quasi-2D Bose gases, the breathing mode can be used to probe the effects of dimensional crossover in the scattering behavior, which manifest itself in the shift of the breathing mode from the universal frequency \(2 \omega_\perp\). The observation of the universal behavior of the breathing mode and its violation are still subjects of intensive interests. It is therefore our hope to communicate the most basic ideas to the readers who are interested. Due to the limitation of the space, many important further developments was not reviewed here. We summarize the relevant references for the convenience of the readers.

This review mainly focused on pure 2D and quasi-2D ultracold Bose gases. Whereas, we want to briefly add that, parallel to Bose gases, precisely the same SO(2, 1) symmetry arises in the case of a unitary Fermi gas in 3D (\(a_{3D} \to \infty\)), which is scale and conformally invariant dictating some universal properties. Along this research line, Castin\textsuperscript{42} has first shown that the time evolution of a unitary gas in an isotropic 3D time-dependent harmonic trap can be exactly given by a gauge and scaling transform. Then, Werner and Castin\textsuperscript{43} proceeded to derive some exact properties of the unitary gas in an isotropic harmonic trap: (i) The spectrum is formed of ladders; the steps of a ladder are spaced by an energy \(2 \hbar \omega_\perp\), and linked by raising and lowering operators. (ii) These properties can be interpreted in terms of a hidden SO(2, 1) symmetry. Furthermore, Mehen\textsuperscript{97} revisited the unitary gas using the state-operator method. Moroz\textsuperscript{47} continued to study the exact time evolution of the density distribution in the position space of the unitary Fermi gas for any spatial dimension, as well as discuss the breathing mode and an alternative for measuring the Tan contact. Vogt et al.\textsuperscript{96} experimentally investigated the collective excitations.
of a harmonically trapped 2D Fermi gas from the collisionless to the hydrodynamic regime; and the breathing mode was observed with an undamped amplitude at a frequency 2 times the dipole mode frequency for a large range of interaction strengths and different temperatures. This experiment provides a direct evidence for a dynamical SO(2, 1) scale symmetry of the 2D Fermi gas. Hofmann and Taylor have respectively showed that the classical SO(2, 1) symmetry of a harmonically trapped 2D Fermi gas can be broken by quantum anomaly. Then Gao and Yu have revisited these results using the sum rule approach. While many theoretical work have predicted the existence of broken SO(2, 1) symmetry in various contexts, so far no experiments show convincing observation of frequency shift of breathing mode from $2\omega_\perp$.

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